SU(2) REPRESENTATIONS AND A LARGE SURGERY FORMULA

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ABSTRACT. A knot $K \subset S^3$ is called $SU(2)$-abundant if for all but finitely many $r \in \mathbb{Q}\backslash\{0\}$, there is an irreducible representation $\pi_1(S_r^3(K)) \to SU(2)$, and the slope $r = u/v \neq 0$ with no irreducible $SU(2)$ representation must satisfy $\Delta_K(\zeta^2) = 0$ for some u-th root of unity ζ. We prove that a nontrivial knot $K \subset S^3$ is $SU(2)$-abundant unless it is a prime knot and the coefficients of its Alexander polynomial $\Delta_K(t)$ lie in $\{-1, 0, 1\}$. In particular, any hyperbolic alternating knot is $SU(2)$-abundant. The proof is based on a large surgery formula that relates instanton knot homology $KHI(S^3, K)$ and the framed instanton homology $I^\sharp(S_n^3(K))$ for any integer n satisfies $|n| \geqslant 2g(K) + 1$. By the same technique, we can calculate many examples of instanton Floer homology. First, for any Berge knot K, the spaces $KHI(S^3, K)$ and $\widehat{HFK}(S^3, K)$ have the same dimension. Second, for any dual knot $K_r \subset S_r^3(K)$ of a Berge knot K with $r > 2g(K) - 1$, we show $\dim_{\mathbb{C}} KHI(S_r^3(K), K_r) = |H_1(S_r^3(K); \mathbb{Z})|$. Third, for any genus-one alternating knot K and any $r \in \mathbb{Q}\backslash\{0\}$, the spaces $I^\sharp(S_r^3(K))$ and $\widehat{HF}(S_r^3(K))$ have the same dimension.

CONTENTS

1. Introduction

The fundamental group is the most important invariant of a 3-manifold. Since it is usually hard to understanding the fundamental group directly, studying homomorphisms from the fundamental group to simpler groups (like $SU(2), SL(2,\mathbb{C}), SL(2,\mathbb{R})$) is a fruitful approach to obtaining computable invariants. For example, the Casson invariant [AM90] and the Casson-Lin invariant [Lin92] are constructed using $SU(2)$ representations, and the A-polynomial [CCG$^+$94] is constructed using the $SL(2,\mathbb{C})$ character variety.

In this paper, we study $SU(2)$ representations of a 3-manifold Y, *i.e.* homomorphisms from the fundamental group $\pi_1(Y)$ to $SU(2)$. For a knot K in S^3, let $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ denote its Alexander polynomial with conditions

$$(1.1) \qquad\qquad \Delta_K(t) = \Delta_K(t^{-1}) \text{ and } \Delta_K(1) = 1.$$

For a knot K in a closed 3-manifold Y, we write $Y(K) = Y \backslash \text{int} N(K)$ for the knot complement and $Y_r(K)$ for the manifold obtained from Y by Dehn surgery along K with slope r in some basis of $H_1(\partial Y(K); \mathbb{Z})$. If K is null-homologous, then we use the meridian and the Seifert longitude of K as a canonical basis of $H_1(\partial Y(K); \mathbb{Z})$.

Definition 1.1. An $SU(2)$ representation is called **abelian** if the image is contained in an abelian subgroup of $SU(2)$. An $SU(2)$ representation is called **irreducible** if it is not abelian. A knot $K \subset S^3$ is called $SU(2)$-**abundant** if the following two conditions hold.

(1) For all but finitely many $r \in \mathbb{Q}\backslash\{0\}$, the manifold $S_r^3(K)$ has an irreducible $SU(2)$ representation.
(2) For any $r = u/v \neq 0$ so that $S_r^3(K)$ has only abelian $SU(2)$ representations, there is some u-th root of unity ζ so that $\Delta_K(\zeta^2) = 0$.

Remark 1.2. The first condition implies K is not $SU(2)$-**averse** in the sense of [SZ20]. Note that if $b_1(Y) = 0$, then an $SU(2)$ representation of Y has abelian image if and only if it has cyclic image. The second condition corresponds to some nondegenerate condition in [BS18b, Corollary 4.8]. By [BS19, Remark 1.6], when u is a prime power, $\Delta_K(\zeta^2) \neq 0$ for any K and any u-th root of unity ζ. Moreover, rationals with prime power numerators are dense in $\mathbb{Q}$.

Suppose $K \subset S^3$ is a nontrivial knot and $r \in \mathbb{Q}$. It is already known that if $|r| \leqslant 2$ [KM04a, Theorem 1] or $|r|$ is sufficiently large [SZ20, Corollary 1.2], then $S_r(K)$ has an irreducible $SU(2)$ representation. There are many other closed 3-manifolds with irreducible $SU(2)$ representations; see [KM04b, Lin16, Zen17, Zen18, BS18b, LPCZ21, BS21, SZ21, XZ21].

In this paper, we provide some sufficient conditions for $SU(2)$-abundant knots.

Theorem 1.3. *A nontrivial knot K is $SU(2)$-abundant unless all following conditions hold.*

(1) There exists $k \in \mathbb{N}_+$ and integers $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$ so that

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j}).$$

(2) The Seifert genus satisfies $g(K) = n_k = n_{k-1} + 1$.
(3) K is a prime knot, i.e., it is not a connected sum of two knots.

Remark 1.4. By term (1) and term (2) in Theorem 1.3, we have

$$\text{(1.2)} \qquad \det(K) = |\Delta_K(-1)| \leqslant 2k + 1 \leqslant 2g(K) + 1.$$

Remark 1.5. In [BS19, Theorem 1.5] and [BS20a, Corollary 1.7, and Proposition 5.4], Baldwin and Sivek proved that a nontrivial knot K is $SU(2)$-abundant unless K is both fibred and strongly quasi-positive (up to mirror), the 4-ball genus $g_4(K)$ equals to $g(K)$, and the slope r with no irreducible $SU(2)$ satisfies $|r| \geqslant 2g(K) - 1$. It is worth mentioning that by techniques developed in this paper, it is possible to provide alternative proofs of those results.

From classification results in [OS05b, BM18, LV21], we have the following corollary.

Corollary 1.6. *The following knots are $SU(2)$-abundant.*
(1) Hyperbolic alternating knots, i.e., alternating knots that are not torus knots $T(2, 2n + 1)$.
(2) Montesinos knots (including all pretzel knots), except torus knots $T(2, 2n + 1)$, pretzel knots $P(-2, 3, 2n + 1)$ for $n \in \mathbb{N}_+$ and their mirrors.
(3) Knots that are closures of 3-braids, except twisted torus knots $K(3, q; 2, p)$ with $pq > 0$ and their mirrors, where $K(3, q; 2, p)$ is the closure of a 3-braid made up of a $(3, q)$ torus braid with p full twist(s) on two adjacent strands.

The proof of Theorem 1.3 is based on instanton knot homology $KHI(Y, K)$ [KM10b] and framed instanton homology $I^\sharp(Y)$ [KM11], which are vector spaces over $\mathbb{C}$ for a knot K in a closed 3-manifold Y. There relative $\mathbb{Z}_2$-gradings on $KHI(Y, K)$ and $I^\sharp(Y)$ and a Seifert surface S of K induces a $\mathbb{Z}$-grading on $KHI(Y, K)$ [Li19, GL19], which we write as

$$KHI(Y, K) = \bigoplus_{i \in \mathbb{Z}} KHI(Y, K, S, i).$$

We write $(-Y, K)$ for the induced knot in the manifold $-Y$ obtained from Y by reversing the orientation and call it the **mirror** of K or (Y, K). For a knot K in S^3, we write $\bar{K}$ for the mirror of K, *i.e.* $(S^3, \bar{K}) = (-S^3, K)$. We write $-K$ for the knot with reverse orientation, which is different from $\bar{K}$. Then we have canonical isomorphisms

$$\text{(1.3)} \qquad KHI(-Y, K, S, i) \cong \operatorname{Hom}_{\mathbb{C}}(KHI(Y, K, S, -i), \mathbb{C}) \text{ and } I^\sharp(-Y) \cong \operatorname{Hom}_{\mathbb{C}}(I^\sharp(Y), \mathbb{C}).$$

Definition 1.7. A rational homology sphere Y is called an **instanton L-space** if $\dim_{\mathbb{C}} I^\sharp(Y) = |H_1(Y; \mathbb{Z})|$. A knot K in an instanton L-space Y is called an **instanton L-space knot** if a nontrivial surgery on it also gives an instanton L-space. We call K a **positive instanton L-space knot** if a positive surgery on it also gives an instanton L-space.

Remark 1.8. It follows directly from (1.3) that Y is an instanton L-space if and only if $-Y$ is an instanton L-space. Since $S_r^3(\bar{K}) = -S_{-r}^3(K)$, a positive surgery on K giving an instanton L-space if and only if a negative surgery on $\bar{K}$ giving an instanton L-space. By [SZ20, Theorem 1.1] and [BS19, Corollary 4.8], if $K \subset S^3$ is not $SU(2)$-abundant, then K is an instanton L-space knot. By [BS18b, Theorem 1.5] and passing to the mirror if necessary, we can further assume that for any sufficiently large integer n, the manifold $S_n^3(K)$ is an instanton L-space.

The following theorem is the main theorem of this paper.

Theorem 1.9. *If $K \subset S^3$ is an instanton L-space knot, then K is a prime knot and there exists $k \in \mathbb{N}$ and integers*

$$n_k > n_{k-1} > \cdots > n_1 > n_0 = 0 > n_{-1} > \cdots > n_{1-k} > n_{-k} \text{ with } n_{-j} = -n_j$$

so that

$$\dim_{\mathbb{C}} KHI(S^3, K, S, i) = \begin{cases} 1 & \text{if } i = n_j \text{ for } j \in [-k, k], \\ 0 & \text{else,} \end{cases}$$

where the $\mathbb{Z}_2$-gradings of the generators of $KHI(S^3, K, S, n_j) \cong \mathbb{C}$ are alternating.

We prove Theorem 1.3 by Theorem 1.9.

Proof of Theorem 1.3. By Remark 1.8, if $K \subset S^3$ is not $SU(2)$-abundant, then K is an instanton L-space knot. Then Theorem 1.9 applies to K and we obtain term (3). Since the space in the top $\mathbb{Z}$-grading of $KHI(S^3, K)$ is one-dimensional, it follows from [KM10b, Section 7] that K is fibred. Then by [BS18a, Theorem 1.7], we know that $\dim_{\mathbb{C}} KHI(S^3, K, S, g(K) - 1) \geqslant 1$, and Theorem 1.9 forces the equality holds. Thus, term (1) and term (2) follow from

$$\sum_{i \in \mathbb{Z}} \chi(KHI(S^3, K, S, i)) \cdot t^i = \pm \Delta_K(t)$$

[Lim09, KM10a], where the sign ambiguity is due to the relative $\mathbb{Z}_2$-grading. □

Theorem 1.9 is an instanton analog of [OS05b, Theorem 1.2] in Heegaard Floer theory due to Ozsváth and Szabó. The key step to prove Theorem 1.9 is to establish an instanton version of the large surgery formula in Heegaard Floer theory. We will explain more details about this strategy in Subsection 1.1. Here we state more applications of techniques developed in this paper.

First, we can compare instanton knot homology of a instanton L-space knot $K \subset Y$ to the knot Floer homology $\widehat{HFK}(Y, K)$ introduced in [OS04a, Ras03], which verifies more examples of [KM10b, Conjecture 7.24]. The main inputs are a generalization of Theorem 1.9, results about Heegaard Floer theory from [OS05b, RR17], and the equation of graded Euler characteristics from [LY20]

$$(1.4) \qquad\qquad \chi_{\mathrm{gr}}(KHI(Y, K)) = \chi_{\mathrm{gr}}(\widehat{HFK}(Y, K)) \in \mathbb{Z}[H]/\pm H,$$

where $H = H_1(Y(K); \mathbb{Z})/\mathrm{Tors}$.

Definition 1.10 ([OS04b, OS05b]). A rational homology sphere Y is called an (Heegaard Floer) L-space if $\dim_{\mathbb{F}_2} \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. A knot K in an L-space Y is called an **(Heegaard Floer) L-space knot** if a nontrivial surgery on it also gives an L-space.

Theorem 1.11. *Suppose $K \subset Y$ is a knot with $H_1(Y(K); \mathbb{Z}) \cong \mathbb{Z}$ and suppose the meridian of K represents q times the generator of $H_1(\partial Y(K); \mathbb{Z})$. Suppose K is both an L-space knot and an instanton L-space knot so that $Y_{u/v}(K)$ is an instanton L-space. If $\gcd(q, v) = 1$, then we have*

$$(1.5) \qquad\qquad \dim_{\mathbb{C}} KHI(Y, K) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y, K).$$

Moreover, when fixing the gradings associated to the Seifert surface S of K properly, we have

$$(1.6) \qquad \dim_{\mathbb{C}} KHI(Y, K, S, i) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y, K, S, i) \leqslant 1 \text{ for any } i \in \mathbb{Z}.$$

Remark 1.12. When $H_1(Y(K); \mathbb{Z})$ has torsion, we can still decompose $KHI(Y, K)$ along elements in $H_1(Y(K); \mathbb{Z})$ as in [LY21a]. However, since this decomposition is not canonical and adapting the proofs to this case is subtle, we leave the discussion in this case to the future. That is why we assume $H_1(Y(K); \mathbb{Z}) \cong \mathbb{Z}$ and $\gcd(q, v) = 1$.

Remark 1.13. From [LPCS20, BS20a], for a knot $K \subset S^3$ that is both an L-space knot and an instanton L-space knot, we have $\dim_{\mathbb{C}} I^{\sharp}(S^3_r(K)) = \dim_{\mathbb{F}_2} \widehat{HF}(S^3_r(K))$ for any $r \in \mathbb{Q}$.

From [ABDS20, Corollary 1.3], a Seifert fibred space is an L-space if and only if it is an instanton L-space. In particular, closed 3-manifolds with elliptic geometry are both L-spaces and instanton L-spaces [OS05b, Proposition 2.3] (or equivalently, with finite fundamental group by the Geometrization theorem; see [KL08]). In particular, S^3, the Poincaré sphere $\Sigma(2, 3, 5)$, and all lens spaces $L(p, q)$ are both L-spaces and instanton L-spaces. From [OS05c, Sca15], double-branched covers of Khovanov-thin knots (in particular, all quasi-alternating knots) are also both L-spaces and instanton L-spaces. Note that when Y is an integral homology sphere in Theorem 1.11, then we have $q = 1$ and hence $\gcd(q, v) = 1$ for any v. Thus, we have the following corollary.

Corollary 1.14. *Suppose K is a knot in $Y = S^3$ or the Poincaré sphere $\Sigma(2, 3, 5)$. If there is some $r \in \mathbb{Q} \backslash \{0\}$ so that $Y_r(K)$ is a Seifert fibred L-space or a double-branched cover of Khovanov-thin knot. Then (1.5) and (1.6) hold.*

Remark 1.15. There are many examples of knots in S^3 and $\Sigma(2, 3, 5)$ that admit lens space surgeries, such as Berge's knots [Ber18] in S^3, Tange's knots [Tan09, Theorem 4.1] in $\Sigma(2, 3, 5)$, Hedden's knots [Hed11] in $\Sigma(2, 3, 5)$ dual to T_R and T_L in lens spaces (see also [Ras07, Bak14, BH20]), Baker's tunnel number two knots [BH20] in $\Sigma(2, 3, 5)$. There are also other twist families of knots admiting Seifert fibred L-space surgeries [Mot16, BM19].

Second, we can relate the knot in the following definition to the framed instanton homology of large surgeries on it. The main input is the large surgery formula introduced in Subsection 1.1. The analog in Heegaard Floer theory was proved in [RR17, Section 3].

Definition 1.16. A knot K in an instanton L-space Y is called an **instanton Floer simple knot** if $\dim_{\mathbb{C}} KHI(Y, K) = \dim_{\mathbb{C}} I^{\sharp}(Y) = |H_1(Y; \mathbb{Z})|$.

Theorem 1.17. *Suppose $K \subset Y$ is a knot with $H_1(Y(K); \mathbb{Z}) \cong \mathbb{Z}$. Suppose the basis of $H_1(\partial Y(K); \mathbb{Z})$ is induced by the meridian of K. Then K is an instanton Floer simple knot if and only if for any $r \in \mathbb{Q}$ with $|r|$ sufficiently large, the manifold $Y_r(K)$ is an instanton L-space.*

Remark 1.18. In [LY20, Theorem 1.8], we proved Theorem 1.17 for simple knots in lens spaces without assuming $H_1(Y(K); \mathbb{Z}) \cong \mathbb{Z}$. The technique there is different from the ones in this paper.

Remark 1.19. From [BS19, Theorem 1.5], we know that if $K \subset S^3$ is a positive instanton L-space knot, then $S^3_r(K)$ is an instanton L-space if and only if $r \geq 2g(K) - 1$. Hence we can apply Theorem 1.17 to the dual knot $K_r \subset S^3_r(K)$ of a Berge knot K with $r \geq 2g(K) - 1$ to obtain that K_r is an instanton Floer simple knot.

Third, we can make some calculations for manifolds obtained from surgeries on genus-one knots. If $K \subset S^3$ with $g(K) = 1$, we can use the large surgery formula introduced in Subsection 1.1 to compute $I^{\sharp}(S^3_r(K))$ when $|r|$ sufficiently large (indeed $|r| \geq 2g(K) + 1 = 3$ is large enough). Furthermore, we can compute $I^{\sharp}(S^3_r(K))$ for any slope r by the concordance invariant $\nu^{\sharp}(K)$ defined by Baldwin and Sivek [BS20a]. In particular, we have the following theorem.

Theorem 1.20. *Suppose K is a genus-one alternating knot. Then for any $r \in \mathbb{Q}\backslash\{0\}$, we have*

$$\dim_{\mathbb{C}} I^{\sharp}(S_r^3(K)) = \dim_{\mathbb{F}_2} \widehat{HF}(S_r^3(K)).$$

Remark 1.21. For genus-one Khovanov-thin knots (in particular, genus-one quasi-alternating knots [KM11, Corollary 1.6]), we can also fix the value $\dim_{\mathbb{C}} I^{\sharp}(S_r^3(K))$ up to the mirror of K; see Section 6 for more details.

1.1. **A large surgery formula in instanton theory.**

In this subsection, we sketch the idea of the proof of Theorem 1.9 and introduce a large surgery formula relates $KHI(S^3, K)$ and $I^{\sharp}(S_n^3(K))$ for any integer n satisfies $|n| \geqslant 2g(K) + 1$. By Remark 1.8, we may assume $S_n^3(K)$ is an instanton L-space for any sufficiently large integer n. However, to apply the proof of [OS05b, Theorem 1.2], we need to recover (at least partially) the following structures in instanton theory.

Fact. Suppose K is a knot in S^3 and $n \in \mathbb{N}_+$. We have the following structures in Heegaard Floer theory [OS04b, OS04a, Ras03].

(1) The decomposition of $\widehat{HF}(S_n^3(K))$ associated to $\mathrm{Spin}^c(S_n^3(K)) \cong \mathbb{Z}_n$:

$$\widehat{HF}(S_n^3(K)) = \bigoplus_{[s] \in \mathbb{Z}_n} \widehat{HF}(S_n^3(K), [s]).$$

(2) The filtration on the Heegaard Floer chain complex $\widehat{CF}(S^3)$ associated to K, which induces a spectral sequence from $\widehat{HFK}(S^3, K)$ to $\widehat{HF}(S^3)$.

(3) The large surgery formula computing $\widehat{HF}(S_n(K), [s])$ for any large integer n and $[s] \in \mathbb{Z}_n$ from the filtrations associated to K and $-K$.

(4) The differential D on the doubly-graded Heegaard Floer chain complex $CFK^{\infty}(S^3, K)$, in particular the fact that $D^2 = 0$.

Since we will use bypass maps based on contact geometry throughout the paper, it is more convenient to use manifolds with reverse orientations. For technical reasons, we replace the notation KHI with $\underline{KHI}$. The constructions below can be generalized to a rationally null-homologous knot in a closed 3-manifold. For simplicity, we only discuss the constructions for a knot K in an integral homology sphere Y and deal with the general case in the main body of the paper. Suppose S is a Seifert surface of K.

The analogy of term (1) can be found in [LY20, Section 4]. We write the decompostion as

$$I^{\sharp}(-Y_{-n}(K)) = \bigoplus_{[s] \in \mathbb{Z}_n} I^{\sharp}(-Y_{-n}(K), [s]).$$

Since there is no explicit construction of the chain complex of $\underline{KHI}(Y, K)$, it is hard to construct the filtration directly. Fortunately, it is possible to recover the spectral sequence and then lift the spectral sequence to a filtered chain complex by algebraic construction. For the analog of term (2), we construct two spectral sequences from $\underline{KHI}(-Y, K)$ to $I^{\sharp}(-Y)$ by two types of bypass maps, and construct two filtered differentials d_+ and d_- on $\underline{KHI}(-Y, K)$ with

$$H(\underline{KHI}(-Y, K), d_+) \cong H(\underline{KHI}(-Y, K), d_-) \cong I^{\sharp}(-Y).$$

For the analog of term (3), we need to introduce the bent complex (*c.f.* Construction 3.21 and Construction 3.30) as follows.

For any integer s, the **bent complex** and the **dual bent complex** are the chain complexes

$$A_s = A_s(-Y, K) := (\underline{KHI}(-Y, K), d_s) \text{ and } A_s^{\vee} = A_s^{\vee}(-Y, K) := (\underline{KHI}(-Y, K), d_s^{\vee}),$$

respectively, where for any element $x \in \underline{KHI}(-Y, K, S, k)$,

$$d_s(x) = \begin{cases} d_+(x) & k > 0, \\ d_+(x) + d_-(x) & k = 0, \\ d_-(x) & k < 0, \end{cases} \text{ and } d_s^\vee(x) = \begin{cases} d_-(x) & k > 0, \\ d_+(x) + d_-(x) & k = 0, \\ d_+(x) & k < 0. \end{cases}$$

Since $d_+ \circ d_+ = d_- \circ d_- = 0$, we have $d_s \circ d_s = d_s^\vee \circ d_s^\vee = 0$. Hence we can consider the homologies $H(A_s)$ and $H(A_s^\vee)$. The proof of following theorem is purely algebraic. The main ingredient is the octahedral axiom (TR4) for a triangulated category.

Theorem 1.22 (Large surgery formula). *For a fixed integer n satisfying $|n| \geqslant 2g(K) + 1$, suppose*

$$s_{min} = -|n| + 1 + g(K) \text{ and } s_{max} = |n| - 1 - g(K).$$

For any integer s', suppose $[s']$ is the image of s' in $\mathbb{Z}_{|n|}$. For any integer $s \in [s_{min}, s_{max}]$, we have

$$I^\sharp(-Y_{-n}(K), [s - s_{min}]) \cong \begin{cases} H(A_{-s}) & \text{if } n > 0, \\ H(A_{-s}^\vee) & \text{if } n < 0. \end{cases}$$

We do not know how to construct the analog of the term (4). However, the proof of [OS05b, Theorem 1.2] only uses the fact that $D^2 = 0$ on some subcomplexes of $CFK^\infty(S^3, K)$. Thus, to obtain a proof of Theorem 1.9, we only need some weaker vanishing results. Since the precise statement is too technical, we only state some byproducts in the next subsection, which are of independent interest for contact geometry.

1.2. **Instanton contact element and Giroux torsion.**

For a contact 3-manifold (N, ξ) with convex boundary and dividing set Γ on ∂N, Baldwin and Sivek [BS16] constructed an instanton contact elemen $\theta(N, \Gamma, \xi)$ that lives in a version of sutured instanton homology $\underline{SHI}(-N, -\Gamma)$ [BS15]. Suppose (Y, ξ') is a closed contact 3-manifold and suppose $(Y(1), \delta, \xi'|_{Y(1)})$ is obtained from (Y, ξ') by removing a 3-ball. Then Baldwin and Sivek defined

$$\theta(Y, \xi') := \theta(Y(1), \delta, \xi'|_{Y(1)}) \in \underline{SHI}(-Y(1), -\delta) = I^\sharp(Y).$$

We have the following theorems for the instanton contact element.

Theorem 1.23. *Suppose (N, ξ) is a contact 3-manifold with convex boundary and dividing set Γ on ∂N. Suppose S is an admissible surface (c.f. Definition 3.2) in (N, Γ) and suppose S_+ and S_- are positive region and negative region of S with respect to ξ, respectively. We write the $\mathbb{Z}$-grading associated to S as*

$$\underline{SHI}(-N, -\Gamma) = \bigoplus_{i \in \mathbb{Z}} \underline{SHI}(-N, -\Gamma, S, i).$$

Then the instanton contact element $\theta(N, \Gamma, \xi)$ lives in

$$\underline{SHI}(-N, -\Gamma, S, \frac{\chi(S_+) - \chi(S_-)}{2}).$$

Definition 1.24. A contact closed 3-manifold (Y, ξ) has **Giroux torsion** if there is an embedding of $(T^2 \times [0, 1], \eta_{2\pi})$ into (Y, ξ), where (x, y, t) are coordinates on $T^2 \times [0, 1] \cong \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1]$ and

$$\eta_{2\pi} = \text{Ker}(cos(2\pi t)dx - sin(2\pi t)dy).$$

Theorem 1.25. *If a closed contact 3-manifold (Y, ξ) has Giroux torsion, then its instanton contact element $\theta(Y, \xi) \in I^\sharp(-Y)$ vanishes.*

Remark 1.26. There is a contact element in Heegaard Floer theory, constructed by Ozsváth and Szabó [OS05a] for closed contact 3-manifolds, and extended by Honda, Kazez, and Matić [HKM09] for contact 3-manifolds with convex boundary. The analog of Theorem 1.23 in Heegaard Floer theory holds by definition of the contact element. The analog of Theorem 1.25 in Heegaard Floer theory was first conjectured by Ghiggini [Ghi06, Conjecture 8.3], and then proved by Ghiggini, Honda, and Van Horn-Morris [GHVHM08]. More proofs can be found in [Mas12, Mat13].

Organization. The paper is organized as follows. In Section 2, we collect some algebraic results about spectral sequences and the triangulated category, which are used in the proof of the large surgery formula. In Section 3, we constructed differentials d_+ and d_- on $\underline{\mathrm{KHI}}(Y, K)$ for a rationally null-homologous knot K in a closed 3-manifold Y and prove a generalization of Theorem 1.22. In Section 4, we prove some vanishing results about contact elements and cobordism maps associated to contact structures. In particular, we prove Theorem 1.23 and Theorem 1.25. In Section 5, we use results in former sections to prove a generalization of Theorem 1.3. Moreover, we prove Theorem 1.11 and Theorem 1.17. In Section 6, we study surgeries on genus-one knots in S^3 and prove Theorem 1.20. In Section 7, we provide examples of $SU(2)$-abundant knots and prove Corollary 1.6. In Section 8, we discuss some further directions of techniques introduced in this paper and make some conjectures.

Convention. If it is not mentioned, all manifolds are smooth, oriented, and connected. All contact structures are oriented and positively co-oriented. Homology groups and cohomology groups are with $\mathbb{Z}$ coefficients. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{F}_2$ for the field with two elements.

A knot $K \subset Y$ is called **null-homologous** if it represents the trivial homology class in $H_1(Y; \mathbb{Z})$, while it is called **rationally null-homologous** if it represents the trivial homology class in $H_1(Y; \mathbb{Q})$.

For any compact 3-manifold M, we write $-M$ for the manifold obtained from M by reversing the orientation. For any surface S in a compact 3-manifold M and any suture $\gamma \subset \partial M$, we write S and γ for the same surface and suture in $-M$, without reversing their orientations. For a knot K in a 3-manifold Y, we write $(-Y, K)$ for the induced knot in $-Y$ with induced orientation, called the **mirror knot** of K. The corresponding balanced sutured manifold is $(-Y(K), -\gamma_K)$.

2. Algebraic preliminaries
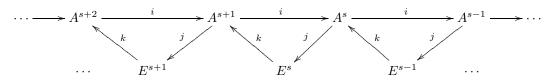
In this section, we collect some algebraic results from homological algebra. All vector spaces are finite-dimensional and over a fixed field.

2.1. **Unrolled exact couples.**

In this subsection, we explain the construction of the spectral sequence from an unrolled exact couple [Boa99] and describe the relationship between the spectral sequence and the filtered chain complex.

Definition 2.1. An **unrolled exact couple** (E^s, A^s) is a diagram of graded vector spaces and homomorphisms of the form

$$\cdots \longrightarrow A^{s+2} \xrightarrow{\quad i \quad} A^{s+1} \xrightarrow{\quad i \quad} A^s \xrightarrow{\quad i \quad} A^{s-1} \longrightarrow \cdots$$

with k, j maps to E^{s+1}, E^s, E^{s-1}

in which each triangle

$$\cdots \to A^{s+1} \to A^s \to E^s \to A^{s+1} \to \cdots$$

is a long exact sequence. An unrolled exact couple is called **bounded** by an interval $[s_1, s_2]$ if $E^s = 0$ for $s \notin [s_1, s_2]$. A morphism between two unrolled exact couples (E^s, A^s) and $(\bar{E}^s, \bar{A}^s)$ consists of maps $f^s : E^s \to \bar{E}^s$ and $g^s : A^s \to \bar{A}^s$ that make all square commute.

Suppose (E^s, A^s) is an unrolled exact couple. For any integers s and r, define

$$\mathrm{Ker}^r A^s = \mathrm{Ker}(i^{(r)} : A^s \to A^{s-r}) \text{ and } \mathrm{Im}^r A^s = \mathrm{Im}(i^{(r)} : A^{s+r} \to A^s),$$

where $i^{(r)}$ denotes the r-fold iterate of i. There are subgroups of E^s:

$$0 = B_1^s \subset B_2^s \subset \cdots \subset \mathrm{Im}\, j = \mathrm{Ker}\, k \subset \cdots \subset Z_2^s \subset Z_1^s = E^s,$$

where

$$B_r^s = j(\mathrm{Ker}^{r-1} A^s) \text{ and } Z_r^s = k^{-1}(\mathrm{Im}^{r-1} A^{s+1}).$$

We call B_r^s and Z_r^s the r-th **boundary subgroup** and the r-th **cycle subgroup** of E^s, respectively. We call the quotient

$$E_r^s = Z_r^s / B_r^s$$

the s-component of the r-th **page**. Note that $E_1^s = E^s$. If the unrolled exact couple is bounded by $[s_1, s_2]$, then we call the direct sum

$$E_r = \bigoplus_{s_1}^{s_2} E_r^s$$

the r-th **page**.

Remark 2.2. If the unrolled exact couple (E^s, A^s) is bounded by $[s_1, s_2]$, then for any integers $r_1, r_2 > s_2 - s_1$ and any integer s, we have

$$B_{r_1}^s = B_{r_2}^s, Z_{r_1}^s = Z_{r_2}^s, E_{r_1}^s = E_{r_2}^s = E_\infty^s, \text{ and } E_{r_1} = E_{r_2} = E_\infty.$$

Proposition 2.3 ([Boa99, Section 0]). *Suppose (E^s, A^s) is an unrolled exact couple. For any integers s and r, there exists a well-defined map*

$$d_r^s : E_r^s \to E_r^{s+r}$$

induced by $j \circ (i^{(r-1)})^{-1} \circ k$ such that

$$d_r^{s+r} \circ d_r^s = 0 \text{ and } \mathrm{Ker}\, d_r^s / \mathrm{Im}\, d_r^{s-r} \cong E_{r+1}^s.$$

Equivalently, the set $\{(E_r^s, d_r^s)\}_{r \geq 1}$ forms a spectral sequence. Moreover, a morphism between two unrolled exact couples induces a map between the corresponding spectral sequences.

Boardman studied the convergence of the spectral sequence in Proposition 2.3 carefully, while we only need the special case for bounded unrolled exact couples.

Theorem 2.4 ([Boa99, Theorem 6.1]). *Suppose (E^s, A^s) is an unrolled exact couple bounded by $[s_1, s_2]$. Then by exactness we have*

$$A^{s_1} \cong A^{s_1-1} \cong A^{s_1-2} \cong \cdots \text{ and } A^{s_2+1} \cong A^{s_2+2} \cong A^{s_2+3} \cong \cdots$$

Consider the spectral sequence $\{(E_r, d_r)\}_{r \geqslant 1}$ from Proposition 2.3, where we omit the superscript s to denote the direct sum of all s-components. Then we have the following results.

(1) *If $A^{s_1} = 0$, then $\{(E_r, d_r)\}_{r \geqslant 1}$ converges to $G = A^{s_2+1}$ with filtration $F^s G = \mathrm{Ker}^{s_2+1-s} A^{s_2+1}$ and we have $F^s G / F^{s+1} G \cong E_\infty^s$.*

(2) *If $A^{s_2+1} = 0$, then $\{(E_r, d_r)\}_{r \geqslant 1}$ converges to $G = A^{s_1}$ with filtration $F^s G = \mathrm{Im}^{s-s_1} A^{s_1}$ and we have $F^s G / F^{s+1} G \cong E_\infty^s$.*

Remark 2.5. The results in [LY20, Section 4.5] are special cases of Proposition 2.3 and Theorem 2.4, where we provided explicits proof by diagram chasing.

It is well-known that a filtered chain complex can induce a spectral sequence. Conversely, we may construct a filtered chain complex from a spectral sequence. However, *a priori* we may lose information when passing a filtered chain complex to a spectral sequence, so the reverse procedure is not always canonical. When fixing an inner product on the first page or equivalently fixing a basis, we have the following canonical construction.

Construction 2.6. Suppose (E^s, A^s) is an unrolled exact couple bounded by $[s_1, s_2]$ and suppose $\{(E_r, d_r)\}_{r \geqslant 1}$ is the spectral sequence from Proposition 2.3. Fix an inner product on $E_1^s = E^s$ for all integers s. For simplicity, we omit the superscript s and consider the direct sum E of all E^s.

For any subgroup X of E, there is a canonical isomorphism $E/X \cong X^\perp$, where $X^\perp$ is the orthogonal complement of X under the fixed inner product. From Definition 2.1 and Remark 2.2, there are subgroups of E:

$$0 = B_1 \subset B_2 \subset \cdots B_{s_2-s_1+1} \subset Z_{s_2-s_1+1} \subset \cdots \subset Z_2 \subset Z_1 = E.$$

For $p = 1, \ldots, s_2 - s_1$, define B_r' as the orthogonal complement of B_p in B_{p+1}, define Z_p' as the orthogonal complement of Z_{p+1} in Z_p, and define E_∞' as the orthogonal complement of $B_{s_2-s_1+1}'$ in $Z_{s_2-s_1+1}'$. Then we have

$$E_r = Z_r / B_r \cong \bigoplus_{p=r}^{s_2-s_1} (B_p' \oplus Z_p') \oplus E_\infty',$$

$$\mathrm{Ker}\, d_r = Z_{r+1} / B_r \cong \bigoplus_{p=r+1}^{s_2-s_1} (B_p' \oplus Z_p') \oplus E_\infty' \oplus B_r',$$

$$\mathrm{Im}\, d_r = B_{r+1} / B_r \cong B_r'$$

Hence we can lift $d_r : E_r \to E_r$ to a map

$$d_r' = I \circ d_r \circ P : E \to E,$$

where P and I are the projection and the inclusion, respectively. The only nontrivial part of d_r' is from Z_r' to B_r', so for any $r_1, r_2 \in \{1, \ldots, s_2 - s_1\}$, we have $d_{r_1}' \circ d_{r_2}' = 0$. Hence the summation

$$d = \sum_{r=1}^{s_2-s_1} d_r'$$

is a differential on E, *i.e.* $d^2 = 0$. Moroever, we have

$$H(E, d) \cong E_\infty' \cong E_{s_2-s_1+1} \cong E_\infty.$$

It is straightforward to check that the filtration $F^s E = \bigoplus_{p \geqslant s} E^p$ on (E, d) induces the spectral sequence $\{(E_r, d_r)\}_{r \geqslant 1}$.

2.2. The octahedral axiom.

It is well-known that the derived category of an abelian category is a triangulated category (for example, see [Wei94, Proposition 10.2.4]). In particular, the derived category of the category of vector spaces is triangulated. Graded vector spaces can be regarded as objects in the derived category with trivial differentials. The following theorem is the special case of the octahedral axiom (TR4) of the triangulated category.

Theorem 2.7. *Suppose X, Y, Z, X', Y', Z' are graded vector spaces satisfying long exact sequences*

$$\cdots \to X \xrightarrow{f} Y \xrightarrow{h} Z' \to X\{1\} \to \cdots$$

$$\cdots \to Y \xrightarrow{g} Z \to X' \xrightarrow{l} Y\{1\} \to \cdots$$

$$\cdots \to X \xrightarrow{g \circ f} Z \to Y' \to X\{1\} \to \cdots$$

where $X\{1\}$ denotes the grading shift of X by 1, so do $Y\{1\}$ and $Z\{1\}$. Then we have the fourth long exact sequence

$$\cdots \to Z' \xrightarrow{\psi} Y' \xrightarrow{\phi} X' \xrightarrow{l \circ h\{1\}} Z'\{1\} \to \cdots$$

such that the following diagram commutes

where the arrows come from four long exact sequences.

Sketch of the proof. We regard graded vector spaces as chain complexes with trivial differentials. By the long exact sequences in the assumption, we know that Z', X', Y' are chain homotopic to mapping cones $\mathrm{Cone}(f), \mathrm{Cone}(g), \mathrm{Cone}(g \circ f)$, respectively. Define

$$\psi : Y \oplus X\{1\} \to Z \oplus X\{1\}$$

$$\psi(y, x) \mapsto (g(y), x)$$

and

$$\phi : Z \oplus X\{1\} \to Z \oplus Y\{1\}$$

$$\phi(z, x) \mapsto (z, f\{1\}(x))$$

The map ψ is a chain map from $\mathrm{Cone}(f)$ to $\mathrm{Cone}(g \circ f)$ and the map ϕ is a chain map from $\mathrm{Cone}(g \circ f)$ to $\mathrm{Cone}(g)$. Since the underlying vector space of $\mathrm{Cone}(\psi)$ is $Z \oplus X\{1\} \oplus Y\{1\} \oplus X\{2\}$,

the inclusion $Z \oplus Y\{1\} \rightarrow Z \oplus X\{1\} \oplus Y\{1\} \oplus X\{2\}$ induces a map η from $\mathrm{Cone}(g)$ to $\mathrm{Cone}(\psi)$, which is a chain map and makes the following diagram commutes

$$
\begin{CD}
\mathrm{Cone}(f) @>\psi>> \mathrm{Cone}(g \circ f) @>\phi>> \mathrm{Cone}(g) @>l \circ h\{1\}>> \mathrm{Cone}(f)\{1\} \\
@V=VV @V=VV @V\eta VV @V=VV \\
\mathrm{Cone}(f) @>\psi>> \mathrm{Cone}(g \circ f) @>>> \mathrm{Cone}(\psi) @>>> \mathrm{Cone}(f)\{1\}
\end{CD}
$$

Define

$$\zeta : Z \oplus X\{1\} \oplus Y\{1\} \oplus X\{2\} \rightarrow Z \oplus Y\{1\}$$
$$\zeta(z, x, y, x') \mapsto (z, y + f(x))$$

Then we can check $\zeta \circ \eta$ is the identity map on $\mathrm{Cone}(g)$ and $\eta \circ \zeta$ is chain homotopic to the identity on $\mathrm{Cone}(\psi)$. Hence $\mathrm{Cone}(f), \mathrm{Cone}(g \circ f)$ and $\mathrm{Cone}(g)$ form a long exact sequence. $\qquad\square$

3. Differentials and the large surgery formula

In this section, we provide more details for constructions in Subsection 1.1 and prove Theorem 1.22. Most notations follow from [LY20, Section 4].

3.1. Backgrounds on sutured instanton homology.

In this subsection, we review some basic facts of sutured instanton homology.

Definition 3.1 ([Juh06, Definition 2.2]). A **balanced sutured manifold** (M, γ) consists of a compact 3-manifold M with non-empty boundary together with a closed 1-submanifold γ on ∂M. Let $A(\gamma) = [-1, 1] \times \gamma$ be an annular neighborhood of $\gamma \subset \partial M$ and let $R(\gamma) = \partial M \backslash \mathrm{int}(A(\gamma))$, such that they satisfy the following properties.

(1) Neither M nor $R(\gamma)$ has a closed component.
(2) If $\partial A(\gamma) = -\partial R(\gamma)$ is oriented in the same way as γ, then we require this orientation of $\partial R(\gamma)$ induces the orientation on $R(\gamma)$, which is called the **canonical orientation**.
(3) Let $R_+(\gamma)$ be the part of $R(\gamma)$ for which the canonical orientation coincides with the induced orientation on ∂M from M, and let $R_-(\gamma) = R(\gamma) \backslash R_+(\gamma)$. We require that $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$. If γ is clear in the contents, we simply write $R_\pm = R_\pm(\gamma)$, respectively.

For any balanced sutured manifold (M, γ), Kronheimer and Mrowka [KM10b, Section 7] constructed a $\mathbb{C}$-vector space $SHI(M, \gamma)$ called the **sutured instanton homology** of (M, γ). The construction was based on closures of (M, γ), *i.e.* a tuple (Y, R, ω) consists of a closed 3-manifold Y, a closed surface $R \subset Y$, and a 1-cycle $\omega \subset Y$ with some admissible conditions.

A priori, the space $SHI(M, \gamma)$ only represents an isomorphism class. Later, Baldwin and Sivek [BS15, Section 9] dealt with the naturality issue and constructed a projectively transitive system **SHI**(M, γ) (twisted version). This system records the collection of vector spaces associated to different closures of (M, γ), which are all isomorphic to $SHI(M, \gamma)$, together with canonical isomorphisms relating these spaces, where these isomorphisms are well-defined up to multiplication by a unit in $\mathbb{C}$.

In practice, when considering maps between sutured instanton homology, we can always fix closures of corresponding balanced sutured manifolds and consider linear maps between actual vector spaces, at the cost that equations between maps only hold up to multiplication by a unit. Hence if it is clear, we will not distinguish the projectively transitive system and the vector space in the system.

To be consistent with notations in [LY20], we write $\underline{SHI}(M,\gamma)$ for the system $\mathbf{\underline{SHI}}(M,\gamma)$. Note that $\underline{SHI}(M,\gamma)$ represents the isomorphism class in [BS15, Section 9], and we write $SHI(M,\gamma)$ for the isomorphism class instead.

There is another projectively transitive system $\mathbf{SHI}^g(M,\gamma)$ (untwisted version) constructed in [BS15, Section 9]. The main difference of two systems is that $\underline{SHI}(M,\gamma)$ corresponds to closures of (M,γ) for which the surface R may have different genera and $\mathbf{SHI}^g(M,\gamma)$ corresponds to closures for which $g = g(R)$ is fixed. Many arguments for $\underline{SHI}(M,\gamma)$ also hold for $\mathbf{SHI}^g(M,\gamma)$ when g is sufficiently large. In [LY21b], we considered $\mathbf{SHI}^g(M,\gamma)$ as a special case of formal sutured homology and calculated its graded Euler characteristic for sufficiently g. By [BS15, Theorem 9], the subsystem of $\underline{SHI}(M,\gamma)$ for closures of fixed genus g is isomorphic to $\mathbf{SHI}^g(M,\gamma)$, so properties of $\mathbf{SH}^g(M,\gamma)$ (especially about graded Euler characteristics) also apply to $\underline{SHI}(M,\gamma)$.

Suppose K is a knot in a closed 3-manifold Y. Let

$$Y(1) := Y\backslash B^3 \text{ and } Y(K) := Y\backslash\text{int}N(K).$$

Suppose δ is a simple closed curve on $\partial Y(K) \cong S^2$ and suppose γ_K is two copies of the meridian of K with opposite orientations. Define

$$I^\sharp(Y) := \underline{SHI}(Y(1),\delta) \text{ and } \underline{KHI}(Y,K) := \underline{SHI}(Y(K),\gamma_K).$$

Note that $I^\sharp(Y)$ also denotes the framed instanton homology of Y constructed in [KM11], though it is isomorphic to $\underline{SHI}(Y(1),\delta)$. So we abuse notation and do not distinguish these two definitions in this paper.

Definition 3.2 ([GL19, Definition 2.25]). Suppose (M,γ) is a balanced sutured manifold and $S \subset (M,\gamma)$ is a properly embedded surface in M. The surface S is called an **admissible surface** if the following conditions hold.

(1) Every boundary component of S intersects γ transversely and nontrivially.
(2) We require that $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is an even integer.

For an admissible surface $S \subset (M,\gamma)$, there is a $\mathbb{Z}$-grading on $\underline{SHI}(M,\gamma)$ [Li19, GL19]:

$$\underline{SHI}(M,\gamma) = \bigoplus_{i\in\mathbb{Z}} \underline{SHI}(M,\gamma,S,i).$$

From the construction of the grading, we have the following basic proposition, which implies (1.3).

Proposition 3.3 ([LY21b, Theorem 2.29]). *For any balanced sutured manifold (M,γ) and any admissible surface $S \subset (M,\gamma)$, there are canonical isomorphisms*

$$SHI(-M,\gamma,S,i) \cong \text{Hom}_\mathbb{C}(SHI(M,\gamma,S,i),\mathbb{C})$$

and

$$SHI(M,\gamma,-S,i) \cong SHI(M,-\gamma,S,i) \cong SHI(M,\gamma,S,-i).$$

3.2. **The caonical basis on the torus boundary.**

In this subsection, we provide a canonical way to fix the basis on the boundary of the knot complement and introduce some notations about sutures.

Suppose Y is a closed 3-manifold and $K \subset Y$ is a null-homologous knot. Let $Y(K)$ be the knot complement $Y\backslash\text{int}(N(K))$. Any Seifert surface S of K gives rise to a framing on $\partial Y(K)$: the longitude λ can be picked as $S \cap \partial Y(K)$ with the induced orientation from S, and the meridian μ can be picked as the meridian of the solid torus $N(K)$ with the orientation so that $\mu \cdot \lambda = -1$. The

'half lives and half dies' fact for 3-manifolds implies that the following map has a 1-dimensional image:
$$\partial_* : H_2(Y(K), \partial Y(K); \mathbb{Q}) \to H_1(\partial Y(K); \mathbb{Q}).$$
Hence any two Seifert surfaces lead to the same framing on $\partial Y(K)$.

Definition 3.4. The framing (μ, λ) defined as above is called the **canonical framing** of (Y, K). With respect to this canonical framing, let
$$\widehat{Y}_{q/p} = Y(K) \cup_\phi S^1 \times D^2$$
be the 3-manifold obtained from Y by a q/p surgery along K, *i.e.*,
$$\phi(\{1\} \times \partial D^2) = q\mu + p\lambda.$$

We also write $\widehat{Y}_\alpha$ for $\widehat{Y}_{q/p}$, where $\alpha = \phi(\{1\} \times \partial D^2)$. When the surgery slope is understood, we also write $\widehat{Y}_{q/p}$ simply as $\widehat{Y}$. Let $\widehat{K}$ be the dual knot, *i.e.*, the image of $S^1 \times \{0\} \subset S^1 \times D^2$ in $\widehat{Y}$ under the gluing map.

Convention. Throughout this section, we will always assume that $\gcd(p, q) = 1$ and $q > 0$ or $(p, q) = (1, 0)$ for a Dehn surgery. Especially, the original pair (Y, K) can be thought of as a pair $(\widehat{Y}, \widehat{K})$ obtained from (Y, K) by the $1/0$ surgery. Moreover, we will always assume that the knot complement $Y(K)$ is irreducible. This is because if $Y(K)$ is not irreducible, then $Y(K) \cong Y'(K') \sharp Y''$ for some closed 3-manifold Y', Y'' and a null-homologous knot $K' \subset Y'$. By the connected sum formula [Li18a, Section 1.8], we have
$$\underline{\mathrm{SHI}}(Y(K), \gamma) \cong \underline{\mathrm{SHI}}(Y'(K'), \gamma) \otimes I^\sharp(Y'')$$
for any suture γ. Hence all results hold after tensoring $I^\sharp(Y'')$.

Next, we describe various families of sutures on the knot complement. Suppose $K \subset Y$ is a null-homologous knot and the pair $(\widehat{Y}, \widehat{K})$ is obtained from (Y, K) by a q/p surgery. Note we can identify the complement of $K \subset Y$ with that of $\widehat{K} \subset \widehat{Y}$, *i.e.* $\widehat{Y}(\widehat{K}) = Y(K)$.

On $\partial Y(K)$, there are two framings: One comes from K, and we write longitude and meridian as λ and μ, respectively. The other comes from $\widehat{K}$. Note only the meridian $\hat{\mu}$ of $\widehat{K}$ is well-defined, and by definition, it is $\hat{\mu} = q\mu + p\lambda$.

Definition 3.5. If $p = 0$, then $q = 1$ and $\hat{\mu} = \mu$. We can take $\hat{\lambda} = \lambda$. If $(q, p) = (0, 1)$, then we take $\hat{\lambda} = -\mu$. If $p, q \neq 0$, then we take $\hat{\lambda} = q_0\mu + p_0\lambda$, where (q_0, p_0) is the unique pair of integers so that the following conditions are true.

(1) $0 \leq |p_0| < |p|$ and $p_0 p \leq 0$.
(2) $0 \leq |q_0| < |q|$ and $q_0 q \leq 0$.
(3) $p_0 q - p q_0 = 1$.

In particular, if $(q, p) = (n, 1)$, then $\hat{\lambda} = -\mu$.

For a homology class $x\lambda + y\mu$, let $\gamma_{x\lambda + y\mu}$ be the suture consisting of two disjoint simple closed curves representing $\pm(x\lambda + y\mu)$ on $\partial Y(K)$. Furthermore, for $n \in \mathbb{Z}$, define
$$\widehat{\Gamma}_n(q/p) = \gamma_{\hat{\lambda} - n\hat{\mu}} = \gamma_{(p_0 - np)\lambda + (q_0 - nq)\mu}, \text{ and } \widehat{\Gamma}_\mu(q/p) = \gamma_{\hat{\mu}} = \gamma_{p\lambda + q\mu}.$$
Suppose $(q_n, p_n) \in \{\pm(q_0 - nq, p_0 - np)\}$ such that $q_n \geq 0$.

When emphasizing the choice of $\hat{\mu}$, we also write $\widehat{\Gamma}_n(\hat{\mu})$ and $\widehat{\Gamma}_\mu(\hat{\mu})$. When $\hat{\lambda}$ and $\hat{\mu}$ are understood, we omit the slope q/p and simply write $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$. When $(q, p) = (1, 0)$, we write Γ_n and Γ_μ instead.

Remark 3.6. Since the two components of the suture must be given opposite orientations, the notations $\gamma_{x\lambda+y\mu}$ and $\gamma_{-x\lambda-y\mu}$ represent the same suture on the knot complement $Y(K)$. Our choice makes $q_{n+1} \leqslant q_n$ for $n < -1$ and $q_{n+1} \geqslant q_n$ for $n \geqslant 0$.

3.3. **Bypass maps on the knot complements.**

In this subsection, we review results in [LY20, Section 4] that are useful in this paper.

If $(M, \gamma) = (Y(K), \gamma_{x\lambda+y\mu})$ and S is an admissible surface obtained from a minimal genus Seifert surface (*c.f.* [LY20, Definition 4.12], where we write S^τ for $\tau \in \{0, -1\}$), then we can calculate the maximal and minimal nontrivial gradings explicitly. Note that we assume that $Y(K)$ is irreducible, so the decomposition of $(Y(K), \gamma_{x\lambda+y\mu})$ along S and $-S$ are both taut (*c.f.* [Juh06, Definition 2.6]). Since we will use contact gluing maps later, it is more convenient to consider $(-M, -\gamma)$ instead of (M, γ).

Definition 3.7. For any integer $y \in \mathbb{N}$, define
$$i_{max}^y = \lceil \frac{y-1}{2} \rceil + g(K), \text{ and } i_{min}^y = \lceil -\frac{y-1}{2} \rceil - g(K),$$
where $\lceil x \rceil$ is the minimal integer larger than x. For $\mu = q\mu + p\lambda$ and the suture $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$, define
$$\hat{i}_{max}^n = i_{max}^{q_n}, \hat{i}_{min}^n = i_{min}^{q_n}, \text{ and } \hat{i}_{max}^\mu = i_{min}^q, \hat{i}_{min}^\mu = i_{min}^q.$$

Lemma 3.8 ([LY20, Lemma 4.14]). *Suppose $K \subset Y$ is a null-homologous knot and $\gamma_{x\lambda+y\mu}$ is a suture on $\partial Y(K)$ with $y \geqslant 0$. Suppose further that S is a Seifert surface of K. Then the maximal and minimal nontrivial gradings of $\underline{SHI}(-Y(K), -\gamma_{(x,y)})$ associated to S are i_{max}^y and i_{min}^y, respectively. In particular, the maximal and minimal nontrivial gradings of $\underline{SHI}(-Y(K), -\widehat{\Gamma}_n)$ associated to S are $\hat{i}_{max}^n$ and $\hat{i}_{min}^n$, respectively.*

It is easy to see that
$$\lim_{n \to +\infty} (\hat{i}_{max}^n - \hat{i}_{min}^n) = \lim_{n \to +\infty} (2g(K) + nq - q_0 - 1) = +\infty.$$

However, by following lemmas, there is no more information in $\underline{SHI}(-Y(K), -\widehat{\Gamma}_n)$ when n is large. To see this, we first introduce the bypass exact triangles.

Definition 3.9. Suppose (M, γ) is a balanced sutured manifold and S is an admissible surface in (M, γ). For any $i, j \in \mathbb{Z}$, define
$$\underline{SHI}(M, \gamma, S, i)[j] = \underline{SHI}(M, \gamma, S, i - j).$$
Moreover, let $\underline{SHI}(M, \gamma, S, i)\{1\}$ be obtained from $\underline{SHI}(M, \gamma, S, i)$ by switch the odd and the even relative $\mathbb{Z}_2$-gradings.

Proposition 3.10 ([LY20, Proposition 4.15], see also [Li19, Proposition 5.5]). *Suppose $K \subset Y$ is a null-homologous knot and suppose the pair $(\widehat{Y}, \widehat{K})$ is obtained from (Y, K) by a q/p surgery. Suppose further that the sutures $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$ are defined as in Definition 3.5 and S is a Seifert surface of K. Then the following conditions hold, where all maps are grading preserving.*

(1) For $n \in \mathbb{Z}$ so that $q_{n+1} = q_n + q$, i.e., $n \geqslant 0$, there are two bypass exact triangles:

$$
\begin{array}{ccc}
\underline{SHI}(-Y(K), -\widehat{\Gamma}_n, S)[\hat{i}_{min}^{n+1} - \hat{i}_{min}^n] & \xrightarrow{\psi_{+,n}^n} & \underline{SHI}(-Y(K), -\widehat{\Gamma}_{n+1}, S) \\
\psi_{+,n}^\mu \Big\uparrow & \swarrow \psi_{+,\mu}^{n+1} & \\
\underline{SHI}(-Y(K), -\widehat{\Gamma}_\mu, S)[\hat{i}_{max}^{n+1} - \hat{i}_{max}^\mu] & &
\end{array}
$$

and

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_n, S)[\hat{i}_{max}^{n+1} - \hat{i}_{max}^{n}] \xrightarrow{\psi_{-,n+1}^{n}} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S)$$

with $\psi_{-,n}^{\mu}$ pointing up on the left and $\psi_{-,\mu}^{n+1}$ pointing down-left, and

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S)[\hat{i}_{min}^{n+1} - \hat{i}_{min}^{\mu}]$$

(2) For $n \in \mathbb{Z}$ so that $q_{n+1} = q_n - q$, i.e., $n < -1$, there are two bypass exact triangles:

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_n, S) \xrightarrow{\psi_{+,n+1}^{n}} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S)[\hat{i}_{max}^{n} - \hat{i}_{max}^{n+1}]$$

with $\psi_{+,n}^{\mu}$ and $\psi_{+,\mu}^{n+1}$, and

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S)[\hat{i}_{min}^{n} - \hat{i}_{min}^{\mu}]$$

and

(3.1) $$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_n, S) \xrightarrow{\psi_{-,n+1}^{n}} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S)[\hat{i}_{min}^{n} - \hat{i}_{min}^{n+1}]$$

with $\psi_{-,n}^{\mu}$ and $\psi_{-,\mu}^{n+1}$, and

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S)[\hat{i}_{max}^{n} - \hat{i}_{max}^{\mu}]$$

(3) For $n \in \mathbb{Z}$ so that $q_{n+1} + q_n = q$, i.e., $n = -1$, there are two bypass exact triangles:

(3.2) $$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_n, S)[\hat{i}_{min}^{\mu} - \hat{i}_{min}^{n}] \xrightarrow{\psi_{+,n+1}^{n}} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S)[\hat{i}_{max}^{\mu} - \hat{i}_{max}^{n+1}]$$

with $\psi_{+,n}^{\mu}$ and $\psi_{+,\mu}^{n+1}$, and

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S)$$

and

(3.3) $$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_n, S)[\hat{i}_{max}^{\mu} - \hat{i}_{max}^{n}] \xrightarrow{\psi_{-,n+1}^{n}} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S)[\hat{i}_{min}^{\mu} - \hat{i}_{min}^{n+1}]$$

with $\psi_{-,n}^{\mu}$ and $\psi_{-,\mu}^{n+1}$, and

$$\underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S)$$

Remark 3.11. The maps $\psi_{+,*}^{*}$ and $\psi_{-,*}^{*}$ are called **bypass maps**, which are contact gluing maps induced by bypass attachments on balanced sutured manifolds. The exact triangles in Proposition 3.10 are called **bypass exact triangles**. In this paper, we will omit the definitions and focus on their algebraic properties.

Lemma 3.12 ([LY20, Lemma 4.37]). *For any surgery slope q/p, consider the bypass maps $\psi^*_{+,*}$ and $\psi^*_{-,*}$ in Proposition 3.10. For any $n \in \mathbb{Z}$, we have two commutative diagrams*

(3.4)

$$\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n) \xrightarrow{\psi^n_{-,n+1}} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1})$$

with maps $\psi^n_{+,\mu}$ and $\psi^{n+1}_{+,\mu}$ to

$$\underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_\mu)$$

and

(3.5)

$$\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n) \xrightarrow{\psi^n_{-,n+1}} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1})$$

with maps $\psi^\mu_{+,n}$ and $\psi^\mu_{+,n+1}$ from

$$\underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_\mu)$$

The similar commutative diagrams hold if we switch the roles of $\psi^*_{+,*}$ and $\psi^*_{-,*}$.

In the following lemma, we abuse the notations for bypass maps so they also denote the restrictions on some gradings associated to S.

Lemma 3.13 ([LY20, Lemma 4.18]). *For any $n \in \mathbb{N}$, the map*

$$\psi^n_{+,n+1} : \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n, S, i) \to \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S, i - \hat{i}^n_{min} + \hat{i}^{n+1}_{min})$$

is an isomorphism if $i \leqslant \hat{i}^n_{max} - 2g(K)$. Similarly, the map

$$\psi^n_{-,n+1} : \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n, S, i) \to \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1}, S, i - \hat{i}^n_{max} + \hat{i}^{n+1}_{max})$$

is an isomorphism if $i \geqslant \hat{i}^n_{min} + 2g(K)$.

Lemma 3.14 ([LY20, Lemma 4.22]). *Suppose $n \in \mathbb{N}$ satisfies $q_n \geqslant q + 2g(K)$, and suppose $i, j \in \mathbb{Z}$ with*

$$\hat{i}^n_{min} + 2g(K) \leqslant i, j \leqslant \hat{i}^n_{max} - 2g(K), \text{ and } i - j = q.$$

Then we have

$$\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n, S, i) \cong \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n, S, j).$$

Thus, we can divide $\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n)$ into three parts: the top $2g(K)$ gradings, the middle gradings, and the bottom $2g(K)$ gradings. All parts stabilize by Lemma 3.13 and the spaces in the middle gradings are cyclic by Lemma 3.14. Moreover, by Proposition 3.3, we have a canonical isomorphism

$$\underline{\mathrm{SHI}}(-M, \gamma, S, i) \cong \underline{\mathrm{SHI}}(-M, -\gamma, S, -i).$$

If $\partial M \cong T^2$, we can identify $-\gamma$ with γ, which induces an involution

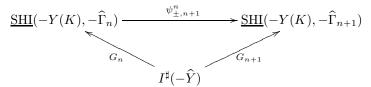(3.6) $$\iota_\gamma : \underline{\mathrm{SHI}}(-M, -\gamma, S, i) \to \underline{\mathrm{SHI}}(-M, \gamma, S, i) \xrightarrow{\cong} \underline{\mathrm{SHI}}(-M, -\gamma, S, -i).$$

Hence the spaces in the top $2g(K)$ gradings and the bottom $2g(K)$ gradings are isomorphic. The following theorems imply that spaces in the middle gradings encode information of $I^\sharp(-\widehat{Y})$.
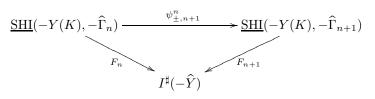
Lemma 3.15 ([LY20, Lemma 4.11], see also [GLW19, Section 3]). *Suppose $K \subset Y$ is a null-homologous knot and suppose the pair $(\widehat{Y}, \widehat{K})$ is obtained from (Y, K) by a q/p surgery. Suppose further that the sutures $\widehat{\Gamma}_n$ are defined as in Definition 3.5. Then, there is an exact triangle*

(3.7)
$$\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n) \xrightarrow{\hspace{4cm}} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1})$$

$$G_n \qquad\qquad\qquad F_{n+1}$$

$$I^\sharp(-\widehat{Y})$$

where F_n is the contact gluing maps associated to the contact 2-handle attachment along $\hat{\mu} = q\mu + p\lambda \subset \partial Y(K)$. Furthermore, we have four commutative diagrams related to $\psi^n_{+,n+1}$ and $\psi^n_{-,n+1}$, respectively

$$\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n) \xrightarrow{\;\;\psi^n_{\pm,n+1}\;\;} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1})$$

$$G_n \qquad\qquad\qquad G_{n+1}$$

$$I^\sharp(-\widehat{Y})$$

and

$$\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n) \xrightarrow{\;\;\psi^n_{\pm,n+1}\;\;} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+1})$$

$$F_n \qquad\qquad\qquad F_{n+1}$$

$$I^\sharp(-\widehat{Y})$$

Theorem 3.16 ([LY20, Proposition 4.28]). *Suppose $n \in \mathbb{N}$ satisfies $q_n \geq q + 2g(K)$. Then there exists an isomorphism*

$$F'_n : \bigoplus_{i=0}^{q-1} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n, S, \hat{i}^n_{max} - 2g(K) - i) \xrightarrow{\;\cong\;} I^\sharp(-\widehat{Y}),$$

where F'_n is the restriction of F_n in Lemma 3.15.

Definition 3.17. For a fixed integer $q > 0$ and any integer $s \in [0, q-1]$, suppose $[s]$ is the image of s in $\mathbb{Z}_q$. Define

$$I^\sharp(-\widehat{Y}, [s]) := F'_n(\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_n, S, \hat{i}^n_{max} - 2g(K) - s)) \subset I^\sharp(-\widehat{Y}).$$

It is well-defined by isomorphisms in Lemma 3.13 and commutative diagrams in Lemma 3.15.

Proposition 3.18 ([LY21b, Corollary 1.20]). *Suppose K is a knot in an integral homology sphere Y and suppose n is an integer. Then $-Y_{-n}(K)$ is an instanton L-space if and only if for any $[s] \in \mathbb{Z}_{|n|}$, we have*

$$\dim_{\mathbb{C}} I^\sharp(-Y_{-n}(K), [s]) = 1.$$

Remark 3.19. Proposition 3.18 also follows from the special case $(M, \gamma) = (Y(1), \delta)$ in [LY21a, Theorem 1.1]:
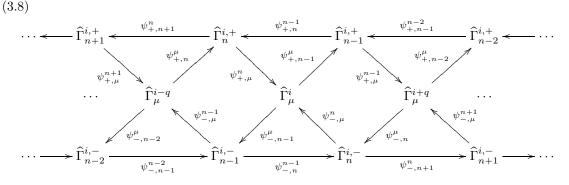
$$\chi_{\mathrm{en}}(I^\sharp(Y)) = \chi(\widehat{HF}(Y)) = \sum_{h \in H_1(Y} h \in \mathbb{Z}[H_1(Y)] / \pm H_1(Y),$$

where Y is any rational homology sphere.

3.4. Two spectral sequences.

In this subsection, we construct spectral sequences from $\underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$ to $I^{\sharp}(-\widehat{Y})$ by bypass exact triangles in Proposition 3.10.

For a fixed integer $q > 0$, any fixed large integer n, and any integer i, we have the following diagram of exact triangles
(3.8)

where we write

$$\widehat{\Gamma}^i_\mu = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, i)$$
$$\widehat{\Gamma}^{i,+}_k = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_k, S, i + \hat{i}^k_{min} - \hat{i}^n_{min} + \hat{i}^n_{max} - \hat{i}^\mu_{max})$$
$$\widehat{\Gamma}^{i,-}_k = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_k, S, i + \hat{i}^k_{max} - \hat{i}^n_{max} + \hat{i}^n_{min} - \hat{i}^\mu_{min})$$

for any $k \in \mathbb{N}$, and we abuse notations so that the maps $\psi^*_{+,*}, \psi^*_{-,*}$ also denote the restrictions on corresponding gradings. Note that $\hat{i}^*_{max}$ and $\hat{i}^*_{min}$ are the maximal and minimal nontrivial gradings of $\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_*)$ associated to S, respectively. By direct calculation, we have

$$(3.9) \qquad \widehat{\Gamma}^{i,+}_{n+k} \cong \widehat{\Gamma}^{i,+}_{n+k-1} \text{ for } k > \frac{i - \hat{i}^\mu_{min}}{q} \text{ and } \widehat{\Gamma}^{i,+}_{n-k} = 0 \text{ for } -k < \frac{i - \hat{i}^\mu_{max}}{q},$$

$$(3.10) \qquad \widehat{\Gamma}^{i,-}_{n+k} \cong \widehat{\Gamma}^{i,-}_{n+k-1} \text{ for } k > \frac{\hat{i}^\mu_{max} - i}{q} \text{ and } \widehat{\Gamma}^{i,-}_{n-k} = 0 \text{ for } -k < \frac{\hat{i}^\mu_{min} - i}{q}.$$

Theorem 3.20. *There exist two spectral sequences* $\{(E_{r,+}, d_{r,+})\}_{r \geq 1}$ *and* $\{(E_{r,-}, d_{r,-})\}_{r \geq 1}$ *with*

$$E_{1,+} = E_{1,-} = \underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$$

induced by exact triangles in (3.8) involving $\psi^*_{+,*}$ *and* $\psi^*_{-,*}$, *respectively. They are independent of the choice of the integer* n. *Suppose* $\{(E_{r,\pm}, d_{r,\pm})\}_{r \geq 1}$ *converge to* $\mathcal{G}_\pm$, *respectively. Then there are isomorphisms*

$$\mathcal{G}_\pm \cong I^{\sharp}(-\widehat{Y}).$$

Proof. The results about the spectral sequences are essentially from [LY20, Section 4.5]. Here we give an alternative proof based on unrolled exact couples introduced in Subsection 2.1.

The exact triangles about $\psi^*_{+,*}$ form an unrolled exact couple in the sense of Definition 2.1. For simplicity, we consider the direct sum of the unrolled exact couples about $i = i_0 + 1, \ldots, i_0 + q$ for some i_0 so that $i \in [\hat{i}^\mu_{min}, \hat{i}^\mu_{max}]$. Then the first page is the same as

$$\underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K}) = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu)$$

Since there are only finitely many nontrivial gradings of associated to S, this unrolled exact couple is bounded. Proposition 2.3 provides a spectral sequence $\{(E_{r,+}, d_{r,+})\}_{r \geqslant 1}$ with $E_{1,+} = \underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$.

Since

$$\hat{i}_{max}^k - \hat{i}_{min}^k = kq - q_0 - 1 + 2g(K) \text{ and } \hat{i}_{max}^\mu - \hat{i}_{min}^\mu = q - 1 + 2g(K),$$

for any integers $i \geqslant \hat{i}_{min}^\mu$ and $k < n - (q - 1 + 2g(K))/q$, we have

(3.11)
$$\begin{aligned}
(i + \hat{i}_{min}^k - \hat{i}_{min}^n + \hat{i}_{max}^n - \hat{i}_{max}^\mu) - \hat{i}_{max}^k &= i + (\hat{i}_{min}^k - \hat{i}_{max}^k) + (\hat{i}_{max}^n - \hat{i}_{min}^n) - \hat{i}_{max}^\mu \\
&= i - (kq - q_0 - 1 + 2g(K)) + (nq - q_0 - 1 + 2g(K)) - \hat{i}_{max}^\mu \\
&= i + (n-k)q - \hat{i}_{max}^\mu \\
&\geqslant \hat{i}_{min}^\mu + (n-k)q - \hat{i}_{max}^\mu \\
&= (n-k)q - (q - 1 + 2g(K)) \\
&> 0.
\end{aligned}$$

For such k, we have $\widehat{\Gamma}_k^{i,+} = 0$. Thus, by Theorem 2.4, we know that $\{(E_{r,+}, d_{r,+})\}_{r \geqslant 1}$ converges to

$$\mathcal{G}_+ = \bigoplus_{i=i_0+1}^{i_0+q} \widehat{\Gamma}_{n+l}^{i,+} \subset \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+l})$$

for some large integer l. The calculation in (3.11) also indicates that $\mathcal{G}_+$ lives in the middle gradings of $\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+l})$. Hence by Lemma 3.14 and Theorem 3.16, we know that $\mathcal{G}_+ \cong I^\sharp(-\widehat{Y})$. The independence of the integer n follows from Lemma 3.13 and Lemma 3.12.

Similar argument applies to exact triangles involving $\psi_{-,*}^*$ and we obtain another spectral sequence $\{(E_{r,-}, d_{r,-})\}_{r \geqslant 1}$ with $E_{1,-} = \underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$, which converges to

$$\mathcal{G}_- \subset \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n+l})$$

in middle gradings for some large integer l. Also, we have $\mathcal{G}_- \cong I^\sharp(-\widehat{Y})$.   □

3.5. Bent complexes.

In this subsection, we construct the bent complex and relate its homology to negative large surgeries. The construction and the name are inspired by Heegaard Floer theory (c.f. [Ras07, Section 4.1], [RR17, Section 2.2]; see also [OS04a, Section 4]).

Construction 3.21. Suppose $\hat{\mu} = q\mu + p\lambda$. Consider the spectral sequences $\{(E_{r,+}, d_{r,+})\}_{r \geqslant 1}$ and $\{(E_{r,-}, d_{r,-})\}_{r \geqslant 1}$ constructed in Theorem 3.20. By fixing a basis of $\underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$, Construction 2.6 provides two filtered chain complexes

$$(\underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K}), d_+) \text{ and } (\underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K}), d_-)$$

such that the induced spectral squences are $\{(E_{r,+}, d_{r,+})\}_{r \geqslant 1}$ and $\{(E_{r,-}, d_{r,-})\}_{r \geqslant 1}$, respectively. For any integer s, the **bent complex** is

$$A_s = A_s(-Y, K) := (\bigoplus_{k \in \mathbb{Z}} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq), d_s),$$

where for any element $x \in \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq)$,

$$d_s(x) = \begin{cases} d_+(x) & k > 0, \\ d_+(x) + d_-(x) & k = 0, \\ d_-(x) & k < 0. \end{cases}$$

It is easy to check $d_s \circ d_s = 0$.

Remark 3.22. Since $\underline{\text{SHI}}$ is a projectively transitive system, the maps $d_{r,+}$ and $d_{r,-}$ only well-defined up to multiplication of a unit. However, the kernel and the image of a map are still well-defined, so we can still define exact sequences for projectively transitive systems. Moreover, if $f : A \to B$ and $g : A \to C$ are maps between projectively transitive systems, though the map

$$f + g := (f, g) : A \to B \oplus C$$

is not well-defined, its kernel $\operatorname{Ker} f \cap \operatorname{Ker} g$ and image $\operatorname{Im} f \oplus \operatorname{Im} g$ are well-defined, so there is no ambiguity to consider the homology of the bent complex. Alternatively, by discussion in Subsection 3.1, we can always fix closures of corresponding balanced sutured manifolds and consider linear maps between actual vector spaces, at the cost that equations between maps only hold up to multiplication by a unit.

The main theorem of this subsection is the following.

Theorem 3.23. *Suppose $\hat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$. For any integer s, let $H(A_s)$ denote the homology of the bent complex A_s in Construction 3.21. For any integer n satisfying $(n-1)q \geqslant 2g(K)$, we have an isomorphism for some integer j_n:*

$$(3.12) \qquad a_{s,n} : H(A_s) \xrightarrow{\cong} \underline{\text{SHI}}(-Y(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, s + j_n).$$

Suppose the maximal and minimal nontrivial gradings of $\underline{\text{SHI}}(-Y(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}})$ are $\hat{i}^\sharp_{max}$ and $\hat{i}^\sharp_{min}$, which can be calculated by Lemma 3.8. Then we have

$$j_n = \hat{i}^\sharp_{min} - \hat{i}^n_{min} + \hat{i}^n_{max} - \hat{i}^\mu_{max} = \hat{i}^\sharp_{max} - \hat{i}^n_{max} + \hat{i}^n_{min} - \hat{i}^\mu_{min}.$$

Remark 3.24. By Definition 3.7, we have $i^y_{max} - i^y_{min} = 2g(K) + y - 1$. Then

$$(\hat{i}^\sharp_{min} - \hat{i}^n_{min} + \hat{i}^n_{max} - \hat{i}^\mu_{max}) - (\hat{i}^\sharp_{max} - \hat{i}^n_{max} + \hat{i}^n_{min} - \hat{i}^\mu_{min})$$

$$= 2(\hat{i}^n_{max} + \hat{i}^n_{min} -) - (\hat{i}^\sharp_{max} - \hat{i}^\sharp_{min}) - (\hat{i}^\mu_{max} - \hat{i}^\mu_{min})$$

$$= 2(nq - q_0 - 1) - ((2n - 1)q - 2q_0 - 1) - (q - 1)$$

$$= 0.$$

Hence j_n in Theorem 3.23 is well-defined.

Proof of Theorem 3.23. We consider two cases. The first case is special, and we use the octahedral axiom to prove it. The second case is more general, and we reduce it to the first case. For the bent complex A_s, we fix $i = s$ in the diagram (3.8).

    Case 1. Suppose $\widehat{\Gamma}^{i,+}_k = \widehat{\Gamma}^{i,-}_k = 0$ for $k \leqslant n - 2$ in the diagram (3.8).

    In this case, higher differentials $d_{r,\pm}$ for $r \geqslant 2$ vanish and the maps

$$\psi^{n-1}_{\pm,\mu} : \widehat{\Gamma}^{i,\pm}_{n-1} \to \widehat{\Gamma}^{i\pm q}_\mu$$

are isomorphisms. Hence

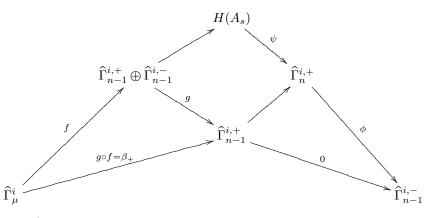$$A_s = (\widehat{\Gamma}^i_\mu \oplus \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1}, f),$$

where

$$f : \widehat{\Gamma}^i_\mu \to \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1}$$
$$f(x) = (\beta_+(x), \beta_-(x))$$

is the restriction of $(\psi^\mu_{+,n-1}(x), \psi^\mu_{-,n-1}(x))$. Define $g : \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1} \to \widehat{\Gamma}^{i,+}_{n-1}$ to be the projection map. Then we apply Theorem 2.7 to

$$X = \widehat{\Gamma}^i_\mu, Y = \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1}, Z = \widehat{\Gamma}^{i,+}_{n-1}, X' = \widehat{\Gamma}^{i,-}_{n-1}, Y' = \widehat{\Gamma}^{i,+}_n, Z' = H(A_s).$$

Then there exist maps ψ and ϕ satisfying the following diagram

Thus, we obtain a long exact sequence

$$\cdots \to H(A_s) \xrightarrow{\psi} \widehat{\Gamma}^{i,+}_n \xrightarrow{\phi} \widehat{\Gamma}^{i,-}_{n-1} \to H(A_s)\{1\} \to \cdots$$

Let

$$\alpha_+ : \widehat{\Gamma}^{i,+}_n \to \widehat{\Gamma}^i_\mu$$

be the restriction of $\psi^n_{+,\mu}$. By the proof of Theorem 2.7, we know that ϕ is constructed by f. Since

$$\widehat{\Gamma}^{i,+}_n \cong \mathrm{Ker}(\beta_+) \oplus \mathrm{Coker}(\beta_+),$$

the map ϕ is zero on $\mathrm{Coker}(\beta_+)$ and the same as $\beta_- \circ \alpha_+$ on $\mathrm{Ker}(\beta_+)$. Thus, we have

(3.13) $$H(A_s) \cong H(\mathrm{Cone}(\phi)) = H(\mathrm{Cone}(\beta_- \circ \alpha_+)).$$

Note that we assume $\hat{\mu} = q\mu + p\lambda$ for $q \geqslant 0$ and $\hat{\lambda} = q_0\mu + p_0\lambda$ satisfying Definition 3.5. When n is large, the coefficient of μ in

$$\hat{\mu}' := n\hat{\mu} - \hat{\lambda} = (nq - q_0)\mu + (np - p_0)\lambda$$

is positive. By Definition 3.5 we set

$$\hat{\lambda}' := \hat{\lambda} - (n-1)\hat{\mu} = (q_0 - (n-1)q)\mu + (p_0 - (n-1)p)\lambda.$$

Then

$$\hat{\lambda}' + \hat{\mu}' = \hat{\mu} \text{ and } \hat{\lambda}' - \hat{\mu}' = 2\hat{\lambda} - (2n-1)\hat{\mu}.$$

Note that $\gamma_{x\lambda+y\mu} = \gamma_{-x\lambda-y\mu}$. Applying the diagram (3.5) with $\psi^-_{+,*}$ and $\psi^+_{-,*}$ switched to

$$\widehat{\Gamma}_\mu(\hat{\mu}') = \gamma_{\hat{\mu}'} = \widehat{\Gamma}_n, \widehat{\Gamma}_{-1}(\hat{\mu}') = \gamma_{\hat{\lambda}'+\hat{\mu}'} = \widehat{\Gamma}_\mu, \text{ and } \widehat{\Gamma}_0(\hat{\mu}') = \gamma_{\hat{\lambda}'} = \widehat{\Gamma}_{n-1},$$

we obtain the following commutative diagram

(3.14) $\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{-1}(\hat{\mu}')) \xrightarrow{\quad\psi_{+,0}^{-1}(\hat{\mu}')\quad} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_0(\hat{\mu}'))$

$\psi_{-,-1}^{\mu}(\hat{\mu}')$ $\psi_{-,0}^{\mu}(\hat{\mu}')$

$\underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_\mu(\hat{\mu}'))$

where the notations $\hat{\mu}'$ in bypass maps indicate that they correspond to $\hat{\mu}'$. By comparing the grading shifts, we have

$$\psi_{+,0}^{-1}(\hat{\mu}') = \beta_- \text{ and } \psi_{-,-1}^{\mu}(\hat{\mu}') = \alpha_+.$$

Indeed, this can be obtained by a diagramatic way in [LY20, Remark 4.17].

Let $\delta : \widehat{\Gamma}_n^{i,+} \to \widehat{\Gamma}_{n-1}^{i,-}$ be the restriction of

$$\psi_{-,0}^{\mu}(\hat{\mu}') : \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_n) \to \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{n-1}).$$

Then (3.14) implies $\delta = \beta_- \circ \alpha_+ = \phi$.

Applying the negative bypass triangle in Theorem 3.10 to

$$\widehat{\Gamma}_\mu(\hat{\mu}') = \gamma_{\hat{\mu}'} = \widehat{\Gamma}_n, \widehat{\Gamma}_0(\hat{\mu}') = \gamma_{\hat{\lambda}'} = \widehat{\Gamma}_{n-1}, \text{ and } \widehat{\Gamma}_1(\hat{\mu}') = \gamma_{\hat{\lambda}'-\hat{\mu}'} = \gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}},$$

we have the following exact triangle

(3.15) $\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_0(\hat{\mu}')) \xrightarrow{\quad\psi_{-,1}^{0}(\hat{\mu}')\quad} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_1(\hat{\mu}'))$

$\psi_{-,0}^{\mu}(\hat{\mu}')$ $\psi_{-,\mu}^{1}(\hat{\mu}')$

$\underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_\mu(\hat{\mu}'))$

By grading shifts in Theorem 3.10, the restriction of (3.15) on a single grading implies

(3.16) $$H(\mathrm{Cone}(\delta)) \cong \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n)$$

Then the isomorphism in (3.12) follows from (3.13) and (3.16).

Case 2. We do not suppose $\widehat{\Gamma}_k^{i,+} = \widehat{\Gamma}_k^{i,-} = 0$ for all $k \leqslant n-2$ in the diagram (3.8). Since $(n-1)q \geqslant 2g(K)$ and $i \in [\hat{i}_{min}^{\mu}, \hat{i}_{max}^{\mu}]$, we have

$$|\frac{i - \hat{i}_{min}^{\mu}}{q}|, |\frac{i - \hat{i}_{max}^{\mu}}{q}| \leqslant |\frac{\hat{i}_{max}^{\mu} - \hat{i}_{min}^{\mu}}{q}| = \frac{q - 1 + 2g(K)}{q} < n.$$

By (3.9) and (3.10), we have $\widehat{\Gamma}_0^{i,\pm} = 0$.

In this case, let

$$A_s' = (\bigoplus_{k \in \mathbb{Z}\backslash\{0\}} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq), d_s)$$

be the subcomplex of A_s. The quotient A_s/A_s' is $\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s)$ with no differentials. Then we have a long exact sequence

$$\cdots \to H(A_s') \to H(A_s) \to H(A_s/A_s') \xrightarrow{\partial_*} H(A_s')\{1\} \to \cdots$$

Since $\widehat{\Gamma}_0^{i,\pm} = 0$, by Theorem 2.4, we know that

(3.17) $$H(A_s') \cong \widehat{\Gamma}_{n-1}^{i,+} \oplus \widehat{\Gamma}_{n-1}^{i,-}.$$

It is straightforward to check $\partial_* = (\beta_+, \beta_-)$ under the isomorphism (3.17). Then by Case 1, we have

$$H(A_s) \cong H(\mathrm{Cone}(\partial_*)) \cong H(\mathrm{Cone}(f)) \cong H(\mathrm{Cone}(\phi)) \cong \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n).$$

$\square$

Then we prove the large surgery formula for negative surgeres.

Theorem 3.25 (Theorem 1.22, $n > 0$). *Suppose $\widehat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$ and suppose $\widehat{\lambda} = q_0\mu + p_0\lambda$ is defined as in Definition 3.4. Note that when $(q, p) = (1, 0)$, we have $(q_0, p_0) = (0, 1)$. For a fixed integer n satisfying $(n-1)q \geqslant 2g(K)$, suppose*

$$\hat{\mu}' = n\hat{\mu} - \hat{\lambda} = (nq - q_0)\mu + (np - p_0)\lambda.$$

For any integer s', suppose $[s']$ is the image of s' in $\mathbb{Z}_{(nq-q_0)}$. Suppose

$$s_{min} = -(nq - q_0 - 1) - \lceil -\frac{q-1}{2} \rceil + g(K) \text{ and } s_{max} = (nq - q_0 - 1) - \lceil \frac{q-1}{2} \rceil - g(K)$$

and suppose an integer $s \in [s_{min}, s_{max}]$. For such n and s, there is an isomorphism

$$H(A_{-s}) \cong I^\sharp(-\widehat{Y}_{\hat{\mu}'}, [s - s_{min}]).$$

Remark 3.26. When $(n-1)q \geqslant 2g(K)$, there are more than $(nq - q_0)$ integers in the interval $[s_{min}, s_{max}]$. Thus, the bent complexes contain all information of $I^\sharp(-\widehat{Y}_{\hat{\mu}'})$.

Proof of Theorem 3.25. Since $(n-1)q \geqslant 2g(K)$, we apply Theorem 3.23 to obtain

$$H(A_{-s}) \cong \underline{\mathrm{SHI}}(-Y(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n - s).$$

We adapt the notations

$$\hat{\lambda}' = \hat{\lambda} - (n-1)\hat{\mu} \text{ and } \hat{\lambda}' - \hat{\mu}' = 2\hat{\lambda} - (2n-1)\hat{\mu} = (2q_0 - (2n-1)q)\mu + (2p_0 - (2n-1)p)\lambda$$

from the proof of Theorem 3.23. Then $\widehat{\Gamma}_1(\hat{\mu}') = \gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}$. Since $(n-1)q \geqslant 2g(K)$, we have

$$(2n-1)q - 2q_0 \geqslant nq - q_0 + 2g(K).$$

Hence we can apply Theorem 3.16 to obtain

$$I^\sharp(-\widehat{Y}_{\hat{\mu}'}, [s]) \cong \underline{\mathrm{SHI}}(-Y(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, \hat{i}^\sharp_{max} - 2g(K) - s).$$

By direct calculation, we have

$$\begin{aligned} j_n - s_{min} &= \hat{i}^\sharp_{max} - \hat{i}^n_{max} + \hat{i}^n_{min} - \hat{i}^\mu_{min} - s_{min} \\ &= \hat{i}^\sharp_{max} - 2g(K) - (nq - q_0 - 1) - \lceil -\frac{q-1}{2} \rceil + g(K) - s_{min} \\ &= \hat{i}^\sharp_{max} - 2g(K). \end{aligned}$$

For any $s \in [s_{min}, s_{max}]$, we have

$$\begin{aligned} j_n - s &= \hat{i}^\sharp_{min} - \hat{i}^n_{min} + \hat{i}^n_{max} - \hat{i}^\mu_{max} - s \\ &= \hat{i}^\sharp_{min} + 2g(K) + (nq - q_0 - 1) - \lceil \frac{q-1}{2} \rceil - g(K) - s \\ &\geqslant \hat{i}^\sharp_{min} + 2g(K). \end{aligned}$$

Thus, the isomorphism follows from Definition 3.17 and Lemma 3.14. $\square$

Finally, we state an instanton analog of [OS08, Theorem 2.3] and [OS11, Theorem 4.1], which is an important step of the proof of the mapping cone formula (*c.f.* Section 8).

Construction 3.27. Following notations in Construction 3.21. For $\circ \in \{+, -\}$, define

$$B_s^\circ = B_s(-Y, K) := (\bigoplus_{k \in \mathbb{Z}} \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq), d_\circ)$$

and define

$$\pi_s^\circ : A_s \to B_s^\circ$$

by

$$\pi_s^+(x) = \begin{cases} x & k > 0, \\ 0 & k \leqslant 0, \end{cases} \text{ and } \pi_s^-(x) = \begin{cases} 0 & k \geqslant 0, \\ 0 & k < 0, \end{cases}$$

where $x \in \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq)$.

Suppose $\hat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$. For n and s in Theorem 3.23, let $H(A_s), H(B_s^+), H(B_s^-)$ be homologies of complexes in Construction 3.21 and let $(\pi_s^+)_*, (\pi_s^-)_*$ denote the induced maps on homologies. Let j_n be the integer in Theorem 3.23 and write $\widehat{\Gamma}^{s,\sharp}$ for

$$\underline{\mathrm{SHI}}(-Y(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n + s).$$

By Theorem 3.23, we have an isomorphism

$$a_{s,n} : H(A_s) \xrightarrow{\cong} \widehat{\Gamma}^{s,\sharp}$$

We use notations in (3.8) and set $i = s$. Let

$$\rho_+ : \widehat{\Gamma}^{s,\sharp} \to \widehat{\Gamma}_n^{s,+}$$

be the restriction of $\psi_{-,\mu}^1(\hat{\mu}')$ in the proof of Theorem 3.23. Choose l as in the proof of Theorem 3.20 so that $\widehat{\Gamma}_{n+l}^{s,+} \subset G_+$. Note that $H(B_s^\pm) = \widehat{\Gamma}_{n+l}^{s,\pm}$ by the proof of Theorem 3.20. Let

$$\Phi_{+,n+l}^n : \widehat{\Gamma}_n^{s,+} \to \widehat{\Gamma}_{n+l}^{s,+}$$

be the composition of $\psi_{+,n+k+1}^{n+k}$ for $k = 0, \ldots, l-1$. Similarly, let

$$\rho_- : \widehat{\Gamma}^{s,\sharp} \to \widehat{\Gamma}_n^{s,-}$$

be the restriction of $\psi_{+,\mu}^1(\hat{\mu}')$ and let

$$\Phi_{-,n+l}^n : \widehat{\Gamma}_n^{s,-} \to \widehat{\Gamma}_{n+l}^{s,-} \subset G_-$$

be the composition of $\psi_{-,n+k+1}^{n+k}$ for $k = 0, \ldots, l-1$.

Proposition 3.28. *Then the following diagram commute*

$$\begin{array}{ccc} H(A_s) & \xrightarrow{(\pi_s^\pm)_*} & H(B_s^\pm) \\ \downarrow{\scriptstyle a_{s,n}} & & \downarrow{\scriptstyle =} \\ \widehat{\Gamma}^{s,\sharp} & \xrightarrow{\Phi_{\pm,n+l}^n \circ \rho_\pm} & \widehat{\Gamma}_{n+l}^{s,\pm}, \end{array}$$

Proof. The proof is straightforward by the proof of Theorem 3.23. $\qquad\square$

Remark 3.29. By direct calculation, the difference of gradings of $\widehat{\Gamma}^{s,+}_{n+l}$ and $\widehat{\Gamma}^{s,-}_{n+l}$ is

$$(\hat{i}^{n+l}_{min} - \hat{i}^n_{min} + \hat{i}^n_{max} - \hat{i}^\mu_{max}) - (\hat{i}^{n+l}_{max} - \hat{i}^n_{max} + \hat{i}^n_{min} - \hat{i}^\mu_{min})$$

$$= -(\hat{i}^{n+l}_{max} - \hat{i}^{n+l}_{min}) + 2(\hat{i}^n_{max} - \hat{i}^n_{min}) - (\hat{i}^\mu_{max} - \hat{i}^\mu_{min})$$

$$= -(n+l)q + q_0 + 2(nq + q_0) - q$$

$$= (n - l - 1)q + q_0.$$

Since $\gcd(q, q_0) = 1$, by Lemma 3.14, the space $\widehat{\Gamma}^{s,+}_{n+l}$ and $\widehat{\Gamma}^{s,-}_{n+l}$ correspond to $I^\sharp(-\widehat{Y}, [s_0 + q_0])$ and $I^\sharp(-\widehat{Y}, [s_0])$ for some integer s_0, respectively. Note that the core knot corresponding to $\hat{\mu} = q\mu + p\lambda$ is isotopic to the curve $q_0\mu + p_0\lambda$ on $\partial Y(K)$.

3.6. Dual bent complexes.

In this subsection, we construct the dual bent complex and relate its homology to large positive surgeries. Proofs are similar to those in Subsection 3.5, so we only point out the difference.

Construction 3.30. Following notations in Construction 3.21. For any integer s, define the **dual bent complex** as

$$A^\vee_s = A^\vee_s(-Y, K) := (\bigoplus_{k \in \mathbb{Z}} \underline{SHI}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq), d^\vee_s),$$

where for any element $x \in \underline{SHI}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq)$,

$$d_s(x) = \begin{cases} d_-(x) & k > 0, \\ d_+(x) + d_-(x) & k = 0, \\ d_+(x) & k < 0. \end{cases}$$

Similar to Theorem 3.23 and Theorem 3.25, we have the following theorems.

Theorem 3.31. *Suppose $\hat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$. For any integer s, let $H(A^\vee_s)$ denote the homology of the bent complex $A^\vee_s$ in Construction 3.30. For any integer n satisfying $(n-1)q \geqslant 2g(K)$, we have an isomorphism for some integer $j^\vee_n$:*

$$(3.18) \qquad a^\vee_{s,n} : H(A^\vee_s) \xrightarrow{\cong} \underline{SHI}(-Y(K), -\gamma_{2\hat{\lambda}+(2n+1)\hat{\mu}}, S, s + j^\vee_n).$$

Suppose the maximal and minimal nontrivial gradings of $\underline{SHI}(-Y(K), -\gamma_{2\hat{\lambda}+(2n+1)\hat{\mu}})$ are $\hat{i}^{\sharp,\vee}_{max}$ and $\hat{i}^{\sharp,\vee}_{min}$, which can be calculated by Lemma 3.8. Then we have

$$j^\vee_n = \hat{i}^{\sharp,\vee}_{max} - \hat{i}^{-n}_{max} + \hat{i}^{-n}_{min} - \hat{i}^\mu_{min} = \hat{i}^{\sharp,\vee}_{min} - \hat{i}^{-n}_{min} + \hat{i}^{-n}_{max} - \hat{i}^\mu_{max}.$$

Theorem 3.32 (Theorem 1.22, $n < 0$). *Suppose $\widehat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$ and suppose $\widehat{\lambda} = q_0\mu + p_0\lambda$ is defined as in Definition 3.4. Note that when $(q, p) = (1, 0)$, we have $(q_0, p_0) = (0, 1)$. For a fixed integer n satisfying $(n-1)q \geqslant 2g(K)$, suppose*

$$\hat{\mu}'' = n\hat{\mu} + \hat{\lambda} = (nq + q_0)\mu + (np + p_0)\lambda.$$

For any integer s', suppose $[s']$ is the image of s' in $\mathbb{Z}_{(nq+q_0)}$. Suppose

$$s^\vee_{min} = -(nq + q_0 - 1) - \lceil -\frac{q-1}{2} \rceil + g(K) \text{ and } s^\vee_{max} = (nq + q_0 - 1) - \lceil \frac{q-1}{2} \rceil - g(K)$$

and suppose an integer $s \in [s^\vee_{min}, s^\vee_{max}]$. For such n and s, there is an isomorphism

$$H(A^\vee_{-s}) \cong I^\sharp(-\widehat{Y}_{\hat{\mu}''}, [s - s^\vee_{min}]).$$

Proof of Theorem 3.31. Instead of using the diagram 3.8, we use the following diagram of exact triangles from Proposition 3.10:

(3.19)

where we write

$$\widehat{\Gamma}^i_\mu = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, i)$$

$$\widehat{\Gamma}^{i,+}_{-k} = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{-k}, S, i + \hat{i}^{-k}_{max} - \hat{i}^{-n}_{max} + \hat{i}^{-n}_{min} - \hat{i}^\mu_{min})$$

$$\widehat{\Gamma}^{i,-}_{-k} = \underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{-k}, S, i + \hat{i}^{-k}_{min} - \hat{i}^{-n}_{min} + \hat{i}^{-n}_{max} - \hat{i}^\mu_{max})$$

for any $k \in \mathbb{N}$, and we abuse notation so that the maps $\psi^*_{+,*}, \psi^*_{-,*}$ also denote the restrictions on corresponding gradings. In this case, we have

(3.20) $\qquad \widehat{\Gamma}^{i,+}_{-n-k} \cong \widehat{\Gamma}^{i,+}_{-n-k-1}$ for $k > \dfrac{\hat{i}^\mu_{max} - i}{q}$ and $\widehat{\Gamma}^{i,+}_{-n+k} = 0$ for $-k < \dfrac{\hat{i}^\mu_{min} - i}{q}$,

(3.21) $\qquad \widehat{\Gamma}^{i,-}_{-n-k} \cong \widehat{\Gamma}^{i,-}_{-n-k-1}$ for $k > \dfrac{i - \hat{i}^\mu_{min}}{q}$ and $\widehat{\Gamma}^{i,-}_{-n+k} = 0$ for $-k < \dfrac{i - \hat{i}^\mu_{max}}{q}$.

By Proposition 2.3 and Theorem 2.4, there exist spectral sequences from

$$\bigoplus_{k\in\mathbb{Z}} \widehat{\Gamma}^{i+kq}_\mu$$

to $\widehat{\Gamma}^{i,+}_{-n-l}$ and $\widehat{\Gamma}^{i,-}_{-n-l}$ for some large l. By Lemma 3.12, those spectral sequences are isomorphic to $\{(E_{r,+}, d_{r,+})\}_{r\geqslant 1}$ and $\{(E_{r,-}, d_{r,-})\}_{r\geqslant 1}$ in Theorem 3.20, hence we can define the dual bent complex by maps in (3.19).

By Definition 3.5, we set

$$\hat{\mu}'' = n\hat{\mu} + \hat{\lambda} \text{ and } \hat{\lambda}'' = -\hat{\mu}.$$

Then

$$\hat{\lambda}'' - \hat{\mu}'' = -\hat{\lambda} - (n+1)\hat{\mu} \text{ and } \hat{\lambda}'' - 2\hat{\mu}'' = -2\hat{\lambda} - (2n+1)\hat{\mu}.$$

Note that $\gamma_{x\lambda+y\mu} = \gamma_{-x\lambda-y\mu}$.

Similar to the proof of Theorem 3.23, we consider two cases and finally obtain that

$$H(A^\vee_i) \cong \mathrm{Cone}(\psi^{-n}_{+,\mu} + \psi^{-n}_{-,\mu} : \widehat{\Gamma}^{i,+}_{-n} \oplus \widehat{\Gamma}^{i,-}_{-n}) \rightarrow \widehat{\Gamma}^i_\mu$$

$$\cong \mathrm{Cone}(\psi^\mu_{-,-n-1} \circ \psi^{-n}_{+,\mu} : \widehat{\Gamma}^{i,+}_{-n} : \widehat{\Gamma}^{i,-}_{-n-1}))$$

$$\cong \underline{\mathrm{SHI}}(-Y(K), -\gamma_{2\hat{\lambda}+(2n+1)\hat{\mu}}, S, i + j^\vee_n).$$

$\square$

Proof of Theorem 3.32. Similar to the proof of Theorem 3.25, the isomorphism follows from Theorem 3.16, Definition 3.17, and Lemma 3.14.   □

The following proposition explains the name of the 'dual bent complex'.

Proposition 3.33. $A_s^\vee(-Y, K)$ *is the dual complex of* $A_{-s}(Y, K)$.

Proof. Suppose $(\bar{Y}, \bar{K}) = (-Y, K)$ is the mirror knot of (Y, K). Note that $(-\bar{Y}, \bar{K}) = (Y, K)$. Suppose S is the Seifert surface of S of K. Then $-S$ is the induced Seifert surface of $\bar{K}$. By Proposition 3.3, we have canonical isomorphisms

$$
\begin{aligned}
\underline{\mathrm{SHI}}(-\bar{Y}(\bar{K}), -\widehat{\Gamma}_n, -S, i) &= \underline{\mathrm{SHI}}(Y(K), -\widehat{\Gamma}_{-n}, -S, i) \\
&\cong \underline{\mathrm{SHI}}(Y(K), -\widehat{\Gamma}_{-n}, S, -i) \\
&\cong \mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{SHI}}(-Y(K), -\widehat{\Gamma}_{-n}, S, -i), \mathbb{C})
\end{aligned}
$$

Then this proposition follows from the fact that both diagram (3.8) and diagram (3.19) can be used to define the bent complex and the dual bent complex.   □

3.7. Grading shifts of differentials.

In this subsection, we study the grading shifts of differentials d_+ and d_+ and relate the bent complex to the dual bent complex. First, it is straightforward to check from the construction that the map d_+ increases the $\mathbb{Z}$-grading and d_- decreases the $\mathbb{Z}$-grading. So we focus on the grading shifts of d_+ and d_- on the relative $\mathbb{Z}_2$-grading.

Convention. Throughout this subsection, 'grading' means the relative $\mathbb{Z}_2$-grading and we set $M = Y(K)$ for a rationally null-homologous knot $K \subset Y$. The bypass map $\psi_{+,*}^*$ and the corresponding negative one $\psi_{-,*}^*$ are from $\underline{\mathrm{SHI}}(-M, -\gamma_1)$ to $\underline{\mathrm{SHI}}(-M, -\gamma_2)$ for some γ_1 and γ_2 consisting of two parallel simple closed curves.

Since all bypass maps are homogeneous (they are constructed by cobordism maps, *c.f.* the proof of [BS18a, Theorem 1.20]), the differentials d_+ and d_- are also homogeneous. To study the grading shifts of d_+ and d_-, we first show that bypass maps $\psi_{+,*}^*$ and $\psi_{-,*}^*$ have the same grading shift. We start with the following lemma.

Lemma 3.34. *Suppose* $\psi_{+,*}^*$ *and* $\psi_{-,*}^*$ *are two bypass maps from* $\underline{\mathrm{SHI}}(-M, -\gamma_1)$ *to* $\underline{\mathrm{SHI}}(-M, -\gamma_2)$ *and suppose* ι_{γ_1} *and* ι_{γ_2} *are involutions defined in (3.6). Then we have*

$$\psi_{-,*}^* = \iota_{\gamma_2} \circ \psi_{-,*}^* \circ \iota_{\gamma_1}.$$

Proof. By construction in [LY20, Section 4.2]), that the bypass arc related to $\psi_{+,*}^*$ on $(Y(K), \gamma_{x\lambda+y\mu})$ is the same as the bypass arc related to $\psi_{-,*}^*$ on $(Y(K), -\gamma_{x\lambda+y\mu})$. Hence we show two compositions of maps are the same.   □

Corollary 3.35. *The involution* ι_γ *induces an isomorphism between spectral sequences* $\{(E_{r,+}, d_{r,+})\}_{r \geqslant 1}$ *and* $\{(E_{r,-}, d_{r,-})\}_{r \geqslant 1}$ *constructed in Theorem 3.20 and hence induces an isomorphism between the chain complexes*

$$(\underline{\mathrm{KHI}}(-Y, K), d_+) \text{ and } (\underline{\mathrm{KHI}}(-Y, K), d_-).$$

Moreover, it induces a canonical identification between A_{-s} *and* A_s.

Lemma 3.36. *Suppose* $\psi_{+,*}^*$ *and* $\psi_{-,*}^*$ *are two bypass maps from* $\underline{\mathrm{SHI}}(-M, -\gamma_1)$ *to* $\underline{\mathrm{SHI}}(-M, -\gamma_2)$. *If* x *is a homogeneous element in* $\underline{\mathrm{SHI}}(-M, -\gamma_1)$, *then* $\psi_{+,*}^*(x)$ *and* $\psi_{-,*}^*(x)$ *are homogeneous elements in* $\underline{\mathrm{SHI}}(-M, -\gamma_2)$ *and they have the same grading.*

Proof. By Lemma 3.34, it suffices to prove that ι_γ preserves the grading for any $\gamma \subset \partial M$. By construction of $\underline{\mathrm{SHI}}(-M, -\gamma)$ in [KM10b, BS15], we can construct a closure (Y', R, ω) of $(-M, -\gamma)$ with $g(R) \geqslant 2$ and take the $(2, 2g(R)-2)$-eigenspace of $(\mu(\mathrm{pt}, \mu(h))$ on $I^\omega(Y')$. It is straightforward to check that $(Y', -R, \omega)$ is a closure of $(-M, \gamma)$. Since γ and $-\gamma$ are isotopic on $\partial M \cong T^2$, there is a diffeomorphism between (Y', R, ω) and $(Y', -R, \omega)$. Moreover, under this diffeomorphism, the involution ι_γ becomes the isomorphism between the $(2, 2g(R)-2)$-eigenspace and the $(2, 2-2g(R))$-eigenspace of $(\mu(\mathrm{pt}, \mu(h))$ on $I^\omega(Y')$. Note that $I^\omega(Y')$ has a $\mathbb{Z}_8$-grading and $\mu(\mathrm{pt})$ and $\mu(R)$ have degree -4 and -2, respectively. Explicitly, the involution sends

$$(v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7) \in I^\omega(Y')$$

to

$$(v_0, v_1, -v_2, -v_3, v_4, v_5, -v_6, -v_7),$$

which preserves the $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}_8$-grading. $\square$

Proposition 3.37. *Suppose d_+ and d_- are differentials on $\underline{\mathrm{KHI}}(-Y, K)$ induced by spectral sequences $\{(E_{r,+}, d_{r,+})\}_{r \geqslant 1}$ and $\{(E_{r,-}, d_{r,-})\}_{r \geqslant 1}$ in Theorem 3.20. For any homogeneous element $x \in \underline{\mathrm{KHI}}(-Y, K)$, the gradings of $d_+(x)$ and $d_-(x)$ are different from the grading of x.*

Proof. We only prove for $d_+(x)$. The proof for $d_-(x)$ is similar. We adapt notations in diagram (3.8). Without loss of generality, suppose $x \in \widehat{\Gamma}^i_\mu$. Consider the projection y of $d_+(x)$ on $\widehat{\Gamma}^{i+kq}_\mu$ for some $k \in \mathbb{N}_+$. By construction of d_+, there exist homogeneous elements $z \in \widehat{\Gamma}^{i,+}_{n-1}$ and $w \in \widehat{\Gamma}^{i,+}_{n-k}$ so that

$$y = \psi^{n-k}_{+,\mu}(w) \text{ and } z = \psi^\mu_{+,n-1}(x) = \psi^{n-2}_{+,n-1} \circ \cdots \circ \psi^{n-k}_{+,n-k+1}(w).$$

By Lemma 3.36, the element

$$z' := \psi^{n-2}_{-,n-1} \circ \cdots \circ \psi^{n-k}_{-,n-k+1}(w)$$

has the same grading as z. By Lemma 3.12, we have

$$\psi^{n-1}_{+,\mu}(z') = y.$$

Define

$$u := \psi^{n-1}_{+,n}(z') \text{ and } u' := \psi^{n-1}_{-,n}(z').$$

By Lemma 3.36, they have the same grading. By 3.12, we have

$$\psi^n_{+,\mu}(u') = y.$$

Let $\mathrm{gr}_2(x)$ denote the grading of x and let $\mathrm{gr}_2(\psi^*_{+,*})$ denote the grading shift of $\psi^*_{+,*}$. Then we have

$$\mathrm{gr}_2(y) - \mathrm{gr}_2(x) = (\mathrm{gr}_2(y) - \mathrm{gr}_2(u')) + (\mathrm{gr}_2(u) - \mathrm{gr}_2(z')) + (\mathrm{gr}_2(z) - \mathrm{gr}_2(x))$$
$$= \mathrm{gr}_2(\psi^n_{+,\mu}) + \mathrm{gr}_2(\psi^{n-1}_{+,n}) + \mathrm{gr}_2(\psi^\mu_{+,n-1})$$
$$= 1,$$

where the last equation follows from the fact that the bypass exact triangle shifts the grading (the bypass exact triangle comes from the surgery exact triangle, *c.f.* the proof of [BS18a, Theorem 1.20]). Since any projection of $d_+(x)$ has different grading from x, the we know that $d_+(x)$ has different grading from x. $\square$

4. Vanishing results about contact elements

In this section, we study contact elements in Heegaard Floer theory and instanton theory. In particular, we prove Theorem 1.23, Theorem 1.25, and a vanishing result for cobordism maps. We only need Corollary 4.16 in the rest sections.

4.1. Contact elements in Heegaard Floer theory.

In this subsection, we review the strategy to prove the vanishing result about Giroux torsion in [GHVHM08].

Suppose (N, ξ) is a contact 3-manifold with convex boundary and dividing set Γ on ∂N. Honda, Kazez, and Matić [HKM09] defined an element $c(N, \Gamma, \xi)$ in sutured Floer homology $SFH(-N, -\Gamma)$, called the **contact element** of (N, ξ). When (N, ξ) is obtained from a closed contact 3-manifold (Y, ξ') by removing a 3-ball, the element

$$c(N, \Gamma, \xi) \in SFH(-N, -\Gamma) \cong \widehat{HF}(-Y)$$

recovers the contact element $c(Y, \xi') \in \widehat{HF}(-Y)$ defined by Ozsváth and Szabó [OS05a].

Consider the Giroux torsion defined in Definition 1.24. We have the following vanishing result.

Theorem 4.1 ([GHVHM08, Theorem 1]). *If a closed contact 3-manifold (Y, ξ) has Giroux torsion, then its contact element $c(Y, \xi) \in \widehat{HF}(-Y)$ vanishes.*

Remark 4.2. The statement of Theorem 4.1 in [GHVHM08] is about $\mathbb{Z}$ coefficient. However, since the naturality of SFH is only proved for $\mathbb{F}_2$ coefficient [JTZ18], the contact element in $\mathbb{Z}$ coefficient is not well-defined. Some progress about the naturality for $\mathbb{Z}$ coefficient is made in [Gar19].

Remark 4.3. There are many partial results and applications of Theorem 4.1. See the introduction of [GHVHM08].

Following the notations in [Hon00, Section 5.2], consider a basic slice $N_0 = (T^2 \times I, \bar{\xi})$ with the dividing set Γ_* on $T^2 \times \{i\}$ for $i = 0, 1$ consisting of two parallel curves of slopes $s_0 = \infty$ and $s_1 = 0$. There are two possible choices of tight structures on N_0 corresponding to two bypasses $\psi_{+,0}^\mu$ and $\psi_{-,0}^\mu$. They are both positively co-oriented but have different orientations. Hence the relative Euler classes differ by signs. Let $\bar{\xi}$ be the tight structure on N_0 corresponding to $\psi_{+,0}^\mu$. Let $N_{\frac{n\pi}{2}}$ be obtained from N_0 by rotating counterclockwise by $\frac{n\pi}{2}$. Note that N_π is the basic slice corresponding to $\psi_{-,0}^\mu$ and $N_{\frac{n\pi}{2} + 2\pi} = N_{\frac{n\pi}{2}}$. Define

$$(N_*, \zeta_1^+) = N_0 \cup N_{\frac{\pi}{2}} \cup N_\pi \cup N_{\frac{3\pi}{2}} \cup N_{2\pi} \text{ and } (N_*, \zeta_1^-) = N_\pi \cup N_{\frac{3\pi}{2}} \cup N_{2\pi} \cup N_{\frac{5\pi}{2}} \cup N_{3\pi}.$$

Then Theorem 4.1 follows from the following three lemmas.

Lemma 4.4 ([GHVHM08, Lemma 5]). *A contact closed 3-manifold (Y, ξ) has Giroux torsion if and only if there exists an embedding of $(N_*, \Gamma_*, \zeta_1^+)$ or $(N_*, \Gamma_*, \zeta_1^+)$ into (Y, ξ).*

Remark 4.5. In the definition of Giroux torsion, there is no condition on the orientation of the contact structure. By construction, the contact structures ζ_1^+ and ζ_1^- differ by orientations. In [GHVHM08], the authors did not deal with these two contact structures separately (*c.f.* the definition of ζ_0 in [GHVHM08]) since the proofs are almost identical. Also, in the original statement of [GHVHM08, Lemma 5], the slopes of dividing set on ∂N_* are -1 and -2, respectively. However, there is a diffeomorphism of $T^2 \times I$ sending the slopes to ∞ and 0, respectively. Note that under this diffeomorphism, the slope ∞ is sent to -1.

Lemma 4.6 ([HKM09, Theorem 4.5]). *Let (Y, ξ) be a closed contact 3-manifold and $N \subset Y$ be a compact submanifold (without any closed components) with convex boundary and dividing set Γ. If $c(N, \Gamma, \xi|_N) = 0$, then $c(Y, \xi) = 0$.*

Lemma 4.7 (From the proof of [GHVHM08, Theorem 1]). *The elements $c(N_*, \Gamma_*, \zeta_1^+)$ and $c(N_*, \Gamma_*, \zeta_1^-)$ vanish.*

4.2. Construction of instanton contact elements.

In [BS16], Baldwin and Sivek constructed a contact invariant in sutured instanton theory which we call the **instanton contact element**. In this subsection, we review the construction and prove Theorem 1.23.

Definition 4.8. Suppose (M, γ) is a balanced sutured manifold. A contact structure ξ on M is said to be **compatible** if ∂M is convex and γ is the dividing set on ∂M.

A **contact handle** is a 3-ball B^3 with the standard tight contact structure. The attachment of B^3 to a balanced sutured manifold (M, γ) is called a **contact i-handle attachment** in following cases:

(1) $i = 0$ when the resulting manifold a disjoint union $(M, \gamma) \cup B^3$.
(2) $i = 1$ when B^3 is attached to (M, γ) along two points on the suture γ.
(3) $i = 2$ when B^3 is attached to (M, γ) along a simple closed curve δ on ∂M with $|\delta \cap \gamma| = 2$.

Suppose (M, γ) is a balanced sutured manifold. Let (M', γ') be the resulting manifold after attaching a contact i-handle. Baldwin and Sivek [BS16, Section 3] constructed a map

$$(4.1) \qquad C : SHI(-M, -\gamma) \to SHI(-M', -\gamma').$$

We sketch the construction as follows.

(1) When $i = 0$ or 1, we can construct the same closure for (M, γ) and (M', γ') and define C to be the identity map.
(2) When $i = 2$, suppose $\delta \subset \partial M$ is the attaching curve of the contact handle. Then a closure of (M', γ') can be obtained from a closure of (M, γ) by performing a 0-surgery along δ, with respect to the framing from ∂M. Then C is induced by the corresponding cobordism between closures.

Suppose $(M, \gamma) \subset (M', \gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact strucutre compatible with $(M' \backslash \mathrm{int} M, \gamma' \cup (-\gamma))$. Based on maps associated to contact handle attachments, we can construct a **contact gluing map**

$$\Phi_\xi : SHI(-M, -\gamma) \to SHI(-M', -\gamma').$$

The first author [Li18b] showed that the contact gluing map is functorial, *i.e.* it is indepedent of the contact handle decompositions and gluing two contact structures induces composite maps.

For a balanced sutured manifold (M, γ) and a compatible contact structure ξ, there are a few ways to decompose ξ [HKM09, BS16].

Partial open book decomposition. A partial open book decomposition is a triple (S, P, h) where S is a compact surface with non-empty boundary, $P \subset S$ a subsurface, and $h : P \to S$ an embedding so that h is the identity on $\partial P \cap \partial S$.

Contact cellular decomposition. A contact cellular decomposition of ξ over (M, γ) is, roughly speaking, a Legendrian graph $\mathcal{K} \subset M$ so that $\partial \mathcal{K} \subset \gamma$ and $M \backslash \mathrm{int} N(\mathcal{K})$ is diffeomorphic to a product $[-1, 1] \times F$ for some surface F withboundary and ξ restricts to the $[-1, 1]$-invariant contact structure on $M \backslash \mathrm{int} N(\mathcal{K}) \cong [-1, 1] \times F$.

Contact handle decomposition. A contact handle decomposition is a decomposition of (M, γ, ξ) into contact 0-, 1-, and 2-handles described above.

These three decompositions can be related to each other as follows.

Suppose we have a contact cellular decomposition, *i.e.*, a Legendrian graph $\mathcal{K} \subset M$ so that $M \backslash \mathrm{int} N(\mathcal{K})$ is a product manifold equipped with the product contact structure. Then $M \backslash \mathrm{int} N(\mathcal{K})$ equipped with the restriction of ξ can be decomposed into a contact 0-handle and a few contact 1-handles. Furthermore, each edge of the Legendrian graph $\mathcal{K}$ corresponds to a contact 2-handle attached along a meridian of the edge. This gives rise to a contact handle decomposition of (M, γ, ξ).

Suppose we have a contact handle decomposition of (M, γ, ξ), we can obtain a partial open book decomposition as follows. All 0- and 1- handle form a product sutured manifold $([-1,1] \times S, \{0\} \times \partial S)$. Suppose 2-handles are attached along curves $\delta_1, ..., \delta_n$. Let $P \subset \{1\} \times S$ be a neighborhood of $(\delta_1 \cup \cdots \cup \delta_n) \cap \{1\} \times S$. Isotope $(\delta_1 \cup \cdots \cup \delta_n) \cap \{-1\} \times S$ through $[-1,1] \times S$ onto $\{1\} \times S$. Let $h : P \to S$ be the embedding so that $h|_{\partial S \cap \partial P}$ is the identity and $\delta_i \cap \{1\} \times S$ is sent to the image of $\delta_i \times \{-1\} \times S$ under the isotopy for $i = 1, \ldots, n$. Then (S, P, h) is a partial open book decomposition of (M, γ, ξ).

Suppose we have a partial open book decomposition (S, P, h) of (M, γ, ξ). We know that $([-1,1] \times S, \{0\} \times \partial S)$ is a product sutured manifold that admits a product contact structure ξ_0. This can be decomposed into a contact 0-handle and a few contact 1-handles. Let $a_1, ..., a_n$ be a collection of disjoint properly embedded arcs on S so that $a_i \subset P$ and $S - (a_1 \cup \cdots \cup a_n)$ retracts to $S - P$. Let δ_i be the union of a_i and $h(a_i)$. Then (M, γ, ξ) is obtained from $([-1,1] \times S, \{0\} \times \partial S, \xi_0)$ by attaching contact 2-handles along all δ_i.

Definition 4.9 ([BS16]). Suppose (M, γ) is a balanced sutured manifold and ξ is a compatible contact structure. Suppose ξ has a partial open book decomposition (S, h, P). Let $\delta_1, ..., \delta_n$ be the attaching curves of the contact 2-handles so that (M, γ, ξ) is obtained from $([-1,1] \times S, \{0\} \times \partial S)$ as above. Suppose the element $\mathbf{1}$ is the generator of

$$\underline{\mathrm{SHI}}(-[-1,1] \times S, -\{1\} \times S) \cong \mathbb{C}.$$

Then the **instanton contact element** of (M, γ, ξ) is

$$\theta(M, \gamma, \xi) := C_{\delta_n} \circ \cdots \circ C_{\delta_1}(\mathbf{1}) \in SHI(-M, -\gamma),$$

where C_{δ_i} is the contact gluing map associated to the contact 2-handle attachment along δ_i.

Theorem 4.10 (Baldwin and Sivek [BS16]). *Suppose (M, γ) is a balanced sutured manifold, and ξ is a compatible contact structure. Then the instanton contact element $\theta(M, \gamma, \xi) \in SHI(-M, -\gamma)$ is independent of the choice of the partial open book decomposition and is well-defined up to a unit. In particular, the non-vanishing of the instanton contact element is an invariant property for the contact structure.*

Then we prove the main theorem of this subsection.

Proof of Theorem 1.23. First, we prove the instanton contact element is homogeneous with respect to the $\mathbb{Z}$-grading of $SHI(-M, -\gamma)$ associated to S. From [HKM09, Theorem 1.1], any triple (M, γ, ξ) admits a contact cell decomposition. Hence there exists a Legendrian graph $\mathcal{K}$, so that $(M \backslash \mathrm{int} N(\mathcal{K}), \xi|_{M \backslash \mathrm{int} N(\mathcal{K})})$ is contactomorphic to $([-1,1] \times F, \xi_0)$ for some surface F with boundary and the product contact structure ξ_0. Let $\delta_1, ..., \delta_n$ be a set of meridians of K, one for each edge of $\mathcal{K}$. Then we can obtain the original ξ on M from $([-1,1] \times F, \xi_0)$ by attaching contact 2-handles along $\delta_1, \ldots, \delta_n$. As discussed above, this gives rise to a contact handle decomposition and hence a

partial open book decomposition. From Definition 4.9, we know that

$$\theta(M, \gamma, \xi) = C_{\delta_n} \circ \cdots \circ C_{\delta_1}(\mathbf{1}) \in \underline{\mathrm{SHI}}(-M, -\gamma),$$

where C_{δ_i} is the contact gluing map associated to the contact 2-handle attachment along δ_i.

Suppose $S \subset (M, \gamma)$ is an admissible surface. We can isotope S so that it intersects $\mathcal{K}$ transversely and disjoint from all δ_i. Write

$$S_{\mathcal{K}} = S \cap (M \backslash \mathrm{int} N(\mathcal{K})).$$

We can consider it as a surface inside the product sutured manifold $([-1,1] \times S, \{1\} \times \partial S)$. Note that $\partial S_{\mathcal{K}} \backslash \partial S$ are all meridians of $\mathcal{K}$ and, by construction, each meridian of $\mathcal{K}$ has two intersections with the dividing set on $\partial(M \backslash \mathrm{int} N(\mathcal{K}))$, which is also identified with

$$\{1\} \times \partial S \subset [-1, 1] \times \{1\} \times S.$$

So $S_{\mathcal{K}}$ is also admissible inside $([-1, 1] \times S, \{1\} \times \partial S)$. Since

$$\underline{\mathrm{SHI}}(-[-1, 1] \times S, -\{1\} \times \partial S) \cong \mathbb{C},$$

we know that there exists $i_0 \in \mathbb{Z}$ so that

$$\mathbf{1} \in \underline{\mathrm{SHI}}(-[-1, 1] \times S, -\{1\} \times \partial S, S_{\mathcal{K}}, i_0).$$

From [LY21b, Proposition 4.6], we know that all maps C_{δ_i} preserve the gradings associated to $S_{\mathcal{K}}$ and S, respectively. Thus, we conclude that

$$\theta(M, \gamma, \xi) = C_{\delta_n} \circ \cdots \circ C_{\delta_1}(\mathbf{1}) \in \underline{\mathrm{SHI}}(-M, -\gamma, i_0).$$

Then we need to figure out i_0. Since $\underline{\mathrm{SHI}}(-[-1, 1] \times S, -\{1\} \times \partial S)$ is one-dimensional, the integer i_0 is determined by its graded Euler characteristic (we fix the closure to resolve the ambiguity of $\pm H$). By [LY21b, Proposition 4.3 and Corollary 3.42] (see also [BS20b, Theorem 3.26]), it suffices to calculate i_0 when replacing $\underline{\mathrm{SHI}}$ by SFH. Note that the contact element of any contact structure ξ compatible with (M, γ) lives in $SFH(-M, -\gamma, \mathfrak{s}_\xi)$, where $\mathfrak{s}_\xi$ is the relative spin^c structure corresponding to ξ. The formula of i_0 then follows from [Hon00, Proposition 4.5]. $\qquad \square$

4.3. Vanishing results about Giroux torsion.

Instanton contact elements share similar properties with the contact elements in SFH. To prove the vanishing result about Giroux torsion for instanton contact element (Theorem 1.25), we need to prove lemmas similar to Lemma 4.6 and Lemma 4.7.

The analog of Lemma 4.6 follows directly from the following proposition.

Proposition 4.11 ([Li18b, Corollary 1.4], see also [BS16, Theorem 1.2]). *Consider the notations as above. If the contact structure ξ on $M' \backslash \mathrm{int} M$ is a restriction of a contact structure ξ' on M', then we have*

$$\Phi_\xi(\theta(M, \gamma, \xi'|_M)) = \theta(M', \gamma', \xi') \in \underline{\mathrm{SHI}}(-M', -\gamma').$$

Corollary 4.12. *Let (Y, ξ) be a closed contact 3-manifold and $N \subset Y$ be a compact submanifold (without any closed components) with convex boundary and dividing set Γ. If $\theta(N, \Gamma, \xi|_N) = 0$, then $\theta(Y, \xi) = 0$.*

The following proposition is the analog of Lemma 4.7.

Proposition 4.13. *The instanton contact elements $\theta(N_*, \Gamma_*, \zeta_1^+)$ and $\theta(N_*, \Gamma_*, \zeta_1^-)$ vanish.*

Proof. Since instanton contact elements share most properties with contact elements, we can apply the proof of Lemma 4.7 with mild changes. We sketch the proof and point out the main difference. For simplicity, we only consider $\theta(N_*, \Gamma_*, \zeta_1^+)$. The proof for $\theta(N_*, \Gamma_*, \zeta_1^-)$ is almost identical.

Take a copy $T_\varepsilon = T^2 \times \{\varepsilon\} \subset \text{int} N_*$ with dividing set consisting two curves of slope ∞. Let L be a Legendrian ruling curve on T_ε with slope -1 (*c.f.* Remark 4.5). The Legendrian curve L has twisting number -1 with respect to the framing coming from T_ε. Let $(N', \Gamma', (\zeta_1^+)')$ be obtained from $(N_*, \Gamma_*, \zeta_1^+)$ by a contact $(+1)$-surgery along L. By [BS16, Theorem 4.6], the cobordism map Φ corresponding to the contact $(+1)$-surgery sending $\theta(N_*, \Gamma_*, \zeta_1^+)$ to $\theta((N', \Gamma', (\zeta_1^+)')) = 0$. By [GHVHM08, Lemma 7], the resulting contact structure $(\zeta_1^+)'$ is overtwisted. Hence by [BS16, Theorem 1.3], we have $\theta((N', \Gamma', (\zeta_1^+)')) = 0$. It remains to show Φ is injective (at least on the subspace generated by $\theta(N_*, \Gamma_*, \zeta_1^+)$).

Write $(N_*, \Gamma_*, \zeta_0^+)$ for N_0. In the proof of Lemma 4.7, by considering the relative spin^c structure, the authors of [GHVHM08] showed that $c(N_*, \Gamma_*, \zeta_0^+)$ and $c(N_*, \Gamma_*, \zeta_1^+)$ lie in the same $\mathbb{F}_2$ summand of $SFH(-N_*, -\Gamma_*) \cong \mathbb{F}_2^4$ (we replace $\mathbb{Z}$-summand by $\mathbb{F}_2$ summand for the naturality issue, *c.f.* Remark 4.2). The contact structure ζ_0^+ and the contact structure $(\zeta_0^+)'$ after the contact $(+1)$-surgery along L can be embedded into S^3 and $S^1 \times S^2$ with standard tight contact structures, respectively, which are both Stein fillable. Then both $c(N_*, \Gamma_*, \zeta_0^+)$ and $c(N', \Gamma', (\zeta_0^+)')$ are non-vanishing. Thus, the map Φ is injective on the $\mathbb{F}_2$ summand generated by $c(N_*, \Gamma_*, \zeta_0^+)$.

For sutured instanton homology, the analog of the (nontorsion) relative spin^c decomposition is the decomposition associated to admissible surfaces, constructed in [GL19, Li19]. We can use two annuli
$$A_0 = S^1 \times \{\text{pt}\} \times I, A_1 = \{\text{pt}\} \times S^1 \times I \subset T^2 \times I$$
to construct the decomposition, where the S^1 factors corresponding to curves of slopes ∞ and 0 parallel to the dividing sets, respectively. Since $|\partial A_i \cap \Gamma_*| = 2$ for $i = 0, 1$, by [LY20, Theorem 2.20] there are only two nontrivial gradings for A_i, corresponding to the sutured manifold decomposition along A_i and $-A_i$. It is straightforward to check that sutured manifold decomposition along $\pm A_0 \cup \pm A_1$ gives a 3-ball with a connected suture, whose $\underline{\text{SHI}}$ is 1-dimensional. Thus,
$$\dim_{\mathbb{C}} \underline{\text{SHI}}(-N_*, -\Gamma_*) = 4.$$

By Proposition 1.23, we know that $\theta(N_*, \Gamma_*, \zeta_1^+)$ and $\theta(N_*, \Gamma_*, \zeta_0^+)$ live in the same grading. Since $\underline{\text{SHI}}$ is 1-dimensional in any nontrivial grading, the elements $\theta(N_*, \Gamma_*, \zeta_1^+)$ and $\theta(N_*, \Gamma_*, \zeta_0^+)$ are linear dependent. By [BS16, Corollary 1.6] and the Stein fillablility, both $\theta(N_*, \Gamma_*, \zeta_0^+)$ and $\theta(N', \Gamma', (\zeta_0^+)')$ are non-vanishing. Then Φ is injective on the subspace generated by $\theta(N_*, \Gamma_*, \zeta_0^+)$, and $\Phi(\theta(N_*, \Gamma_*, \zeta_1^+)) = 0$ implies $\theta(N_*, \Gamma_*, \zeta_1^+) = 0$.

□

Proof of Theorem 1.25. This follows from Lemma 4.4, Corollary 4.12, and Proposition 4.13. Note that Lemma 4.4 is only about contact topology, so we can apply it without change. □

4.4. **Vanishing results about cobordism maps.**

Suppose $(M, \gamma) \subset (M', \gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact strucutre compatible with $(M' \backslash \text{int} M, \gamma' \cup (-\gamma))$. By Corollary 4.12, if
$$\theta(M' \backslash \text{int} M, \gamma' \cup (-\gamma), \xi) = 0,$$
then the contact gluing map Φ_ξ vanishes on the subspace of $\underline{\text{SHI}}(-M, -\gamma)$ generated by instanton contact elements. Indeed, we can prove a stronger result by the functoriality of Φ_ξ. The proof of the following proposition is due to Ian Zemke.

Proposition 4.14. *Suppose* $(M, \gamma) \subset (M', \gamma')$ *is a proper inclusion of balanced sutured manifolds and suppose* ξ *is a contact strucutre compatible with*

$$(M_0, \gamma_0) := (M' \backslash \mathrm{int} M, \gamma' \cup (-\gamma)).$$

If the contact element $\theta(M_0, \gamma_0, \xi)$ *vanishes, then the map* Φ_ξ *vanishes on* $\underline{\mathrm{SHI}}(-M, -\gamma)$.

Proof. We have inclusions

$$(M, \gamma) \subset (M, \gamma) \sqcup (M_0, \gamma_0) \subset (M', \gamma'),$$

where $\sqcup$ denotes the disjoint union. The manifold

$$M' \backslash \mathrm{int}(M \sqcup M_0)$$

is contactomorphic to $\partial M \times I$. Let ξ_0 be the product contact structure on $\partial M \times I$. By the connected sum formula [Li18a, Section 1.8], we have

$$\underline{\mathrm{SHI}}(-M \sqcup (-M_0), -\gamma \sqcup (-\gamma_0)) \cong \underline{\mathrm{SHI}}(-M, -\gamma) \otimes \underline{\mathrm{SHI}}(-M_0, -\gamma_0) \otimes \mathbb{C}^2.$$

By functoriality, the map Φ_ξ is the composition of the following maps

$$\underline{\mathrm{SHI}}(-M, -\gamma) \to \underline{\mathrm{SHI}}(-M, -\gamma) \otimes \underline{\mathrm{SHI}}(-M_0, -\gamma_0) \otimes \mathbb{C}^2 \to \underline{\mathrm{SHI}}(-M', -\gamma')$$
$$x \mapsto x \otimes \theta(M_0, \gamma_0, \xi) \otimes y_0 \mapsto \Phi_{\xi_0}(x \otimes \theta(M_0, \gamma_0, \xi)),$$

where y_0 is a canonical element in $\mathbb{C}^2$. If $\theta(M_0, \gamma_0, \xi) = 0$, then $\Phi_\xi = 0$. $\qquad\square$

Remark 4.15. For a general balanced sutured manifold (M, γ), instanton contact elements do not generate $\underline{\mathrm{SHI}}(-M, -\gamma)$ because the number of tight contact structures compatible with (M, γ) is less than $\dim_{\mathbb{C}} \underline{\mathrm{SHI}}(M, \gamma)$. See [Li19, Section 4.3] and [Hon00] for discussion about contact structures on the solid torus.

The following vanishing result is used in the rest of the paper.

Corollary 4.16. *Suppose* $(M, \gamma) \subset (M', \gamma')$ *is a proper inclusion of balanced sutured manifolds. If*

$$(M' \backslash \mathrm{int} M, \gamma' \cup (-\gamma), \xi) = (N_*, \Gamma_*, \zeta_1^+) \text{ or } (N_*, \Gamma_*, \zeta_1^-)$$

defined in Subsection 4.1, then $\Phi_\xi = 0$.

Proof. This follows from Proposition 4.13 and Proposition 4.14 $\qquad\square$

5. INSTANTON L-SPACE KNOTS

In this section, we study the instanton knot homology of an instanton L-space knot $K \subset Y$. In particular, we prove Theorem 1.9, Theorem 1.11, and Theorem 1.17. For technical reasons, We only deal with the case $H_1(Y(K)) \cong \mathbb{Z}$.

5.1. **The dimension in each grading.**
In this subsection, we prove the following theorem. The main input is the large surgery formula and the vanishing result Corollary 4.16.

Theorem 5.1. *Suppose* Y *is an integral homology sphere with* $I^\sharp(Y) \cong \mathbb{C}$. *Suppose* $K \subset Y$ *is a knot and* S *is the Seifert surface of* K. *If there is a positive integer* n *so that* $Y_{-n}(K)$ *is an instanton L-space, then for any* $i \in \mathbb{Z}$, *we have*

$$\dim_{\mathbb{C}} \underline{\mathrm{KHI}}(-Y, K, S, i) \leqslant 1.$$

Since Y is an integral homology sphere, K is always null-homologous and $\hat{\mu} = \mu, \hat{\lambda} = \lambda$ in Subsection 3.2. By Definition 3.5, we have $(q, p) = (1, 0)$ and $(q_0, p_0) = (0, 1)$. Then we have

$$\widehat{\Gamma}_\mu = \Gamma_\mu = \gamma_\mu, \widehat{\Gamma}_n = \Gamma_n = \gamma_{\lambda - n\mu}.$$

Note that in the proof of Theorem 3.23, an auxiliary slope $\hat{\mu}' = n\hat{\mu} - \hat{\lambda}$ is used. Here we set $\hat{\mu}' = n\mu - \lambda$. Since n is not fixed, this slope is also not fixed.

For simplicity, we write $\gamma_{(x,y)}$ for $\gamma_{x\lambda + y\mu}$ in Definition 3.5. Also, we omit S in the notation $\underline{\text{SHI}}(-Y(K), \gamma, S, i)$ for any γ.

Then we make the following definition.

Definition 5.2. For any integers n and i with $|i| \leqslant g(K)$, define

$$T_{n,i} = \underline{\text{SHI}}(-Y(K), -\Gamma_n, i + \lceil \frac{n-1}{2} \rceil),$$

$$B_{n,i} = \underline{\text{SHI}}(-Y(K), -\Gamma_n, i - 1 + \lceil -\frac{n-1}{2} \rceil).$$

For $i > g(K)$ and any n, define $T_{n,i} = 0$. For $i < -g(K)$ and any n, define $B_{n,i} = 0$.

Remark 5.3. The notations 'T' and 'B' mean 'top' and 'bottom'. If we use the notations after the diagram (3.8) and suppose $g = g(K)$, then for any integers n and i with $|i| \leqslant g(K)$, we have

$$T_{n,i} = \widehat{\Gamma}_n^{i,+} \text{ and } B_{n,i} = \widehat{\Gamma}_{n-1}^{i,-}.$$

By Lemma 3.13, we have

$$\psi_{-,n+1}^n : T_{n,i} \xrightarrow{\cong} T_{n+1,i} \text{ and } \psi_{+,n+1} : B_{n,i} \xrightarrow{\cong} B_{n+1,i}$$

for $n \geqslant 2g(K) + 1$ and $|i| \leqslant g(K)$.

The following proposition follows from the large surgery formula.

Proposition 5.4. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n is an integer so that $n \geqslant 2g(K) + 1$ and $Y_{-n}(K)$ is an instanton L-space. Then we have the following.*

$$\underline{\text{SHI}}(-Y(K), -\gamma_{(2,1-2n)}, i) \cong \begin{cases} T_{n,i-n+1} & n - g \leqslant i \leqslant n - 1 + g \\ \mathbb{C} & -n + g + 1 \leqslant i \leqslant n - g - 1 \\ B_{n,i+n-1} & -n + 1 - g \leqslant i \leqslant -n + g \end{cases}$$

Proof. The isomorphism of the top and bottom $2g$ gradings of $\underline{\text{SHI}}(-Y(K), -\gamma_{(2,1-2n)})$ follows from applying Lemma 3.13 to $\hat{\mu}'$. Since $Y_{-n}(K)$ is an instanton L-space, by (1.3), the manifold $-Y_{-n}(K)$ is also an instanton L-space. The isomorphism of the middle gradings follows from Proposition 3.18, Lemma 3.14, and Theorem 1.22. $\square$

Note that in the proof of Theorem 3.23 (more precisely, in the triangle (3.15)), we have a map $\psi_{-,0}^\mu(\hat{\mu}')$ from the space associated to $\widehat{\Gamma}_n$ to the space associated to $\widehat{\Gamma}_{n-1}$. We write this map as $\psi_{-,n-1}^n$. We also write $\psi_{-,n}^{2n-1}$ and $\psi_{-,2n-1}^{n-1}$ for $\psi_{-,\mu}^1(\hat{\mu}')$ and $\psi_{-,1}^0(\hat{\mu}')$ in (3.15)), respectively. Similarly we write $\psi_{+,n-1}^n, \psi_{+,n}^{2n-1}$, and $\psi_{+,2n-1}^{n-1}$ for maps in the positive bypass triangle. We abuse notation so that bypass maps also denote their restrictions on a single grading. Then the following proposition follows from the vanishing results established in Section 4.

Proposition 5.5. *Suppose $K \subset Y$ is a null homologous knot. For any integer $n \in \mathbb{Z}$ with $n \geqslant 2g(K) + 1$ and any integer i with $|i| \leqslant g(K)$, we have*

$$\psi_{+,n}^{n+1} \circ \psi_{-,n+1}^{n+2} = 0 : T_{n+2,i} \to T_{n,i}$$

and

$$\psi_{-,n}^{n+1} \circ \psi_{+,n+1}^{n+2} = 0 : B_{n+2,i} \to B_{n,i}.$$

Proof. By Remark 5.3, it suffices to prove

$$\Psi_T := \psi_{-,n+3}^{n+2} \circ \psi_{-,n+2}^{n+1} \circ \psi_{-,n+1}^{n} \circ \psi_{+,n}^{n+1} \circ \psi_{-,n+1}^{n+2} = 0 : T_{n+2,i} \to T_{n+3,i}$$

and

$$\Psi_B := \psi_{+,n+3}^{n+2} \circ \psi_{+,n+2}^{n+1} \circ \psi_{+,n+1}^{n} \circ \psi_{-,n}^{n+1} \circ \psi_{+,n+1}^{n+2} = 0 : B_{n+2,i} \to B_{n+3,i}.$$
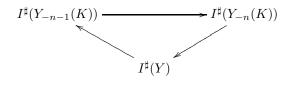
By classification of tight contact structures on $T^2 \times I$ [Hon00], we know that the contact structures corresponding to Ψ_T and Ψ_B are contactomorphic to either $(N_*, \Gamma_*, \zeta_1^+)$ or $(N_*, \Gamma_*, \zeta_1^-)$ defined in Subsection 4.1. Then the lemma follows from Corollary 4.16. $\square$

Proposition 5.6. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n_0 be a positive integer so that $Y_{-n_0}(K)$ is an instanton L-space. Then for any integer n so that $n > n_0$, $Y_{-n}(K)$ is also an instanton L-space.*

Proof. This proposition follows immediately from $\chi(I^\sharp(Y_{-n}(K)) = |H_1(Y_{-n}(K))|$, the equation

$$|H_1(Y_{-n-1}(K))| = |H_1(Y_{-n}(K))| + |H_1(Y)|,$$

and the following surgery exact triangle ([BS18b, Section 4.2], see also [Sca15])

$$I^\sharp(Y_{-n-1}(K)) \longrightarrow I^\sharp(Y_{-n}(K))$$
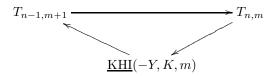$$I^\sharp(Y)$$

$\square$

By Proposition 5.4 and Proposition 5.5, the proof of Theorem 5.1 follows from similar algebraic lemmas in [OS05b, Section 3]. We reprove them in our setting.

Lemma 5.7. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n_0 be a positive integer so that $Y_{-n_0}(K)$ is an instanton L-space. Suppose further that for a large enough integer n and some integer m with $|m| \leqslant g(K)$, we have $T_{n,m+1} = 0$. Then one of the following two cases happens.*

(1) $\underline{KHI}(-Y, K, m) \cong \mathbb{C}$ and $B_{n,m-1} = 0$,
(2) $\underline{KHI}(-Y, K, m) = 0$ and $T_{n,m} = 0$.

Proof. By Proposition 5.6, we can take an arbitrary large enough integer n, since they are all L-space surgery slopes. From Proposition 3.10, we have the following exact triangle

$$T_{n-1,m+1} \longrightarrow T_{n,m}$$
$$\underline{KHI}(-Y, K, m)$$

From Remark 5.3 and the assumption $T_{n,m+1} = 0$, we know that

$$T_{n-1,m+1} \cong T_{n,m+1} = 0 \text{ and } B_{n,m-1} \cong B_{n-1,m-1}.$$

Hence there exists some $k \in \mathbb{N}$ so that

$$T_{n,m} \cong \underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}^k.$$

Also from Proposition 3.10, we have the following exact diagram

$$\underline{\mathrm{SHI}}(-Y(K), -\gamma_{(2,1-2n)}, m)$$

$$\downarrow$$

$$T_{n,m}$$

$$\downarrow \psi^{n,m}_{-,n-1}$$

$$\underline{\mathrm{SHI}}(-Y(K), -\gamma_{(2,3-2n)}, m-1) \xrightarrow{\psi^{2n-3,m-1}_{+,n-1}} B_{n-1,m-1} \xrightarrow{\psi^{n-1,m-1}_{+,n-2}} T_{n-2,m}$$

where $\psi^{n,m}_{-,n-1}$ is the map $\psi^n_{-,n-1}$ restricted to the graded part $T_{n,m}$ and other notations are defined similarly. Since $|m| \leqslant g(K)$, Proposition 5.4 implies that

$$\underline{\mathrm{SHI}}(-Y(K), -\gamma_{(2,1-2n)}, m) \cong \underline{\mathrm{SHI}}(-Y(K), -\gamma_{(2,3-2n)}, m-1) \cong \mathbb{C}.$$

Hence the above diagram can be re-write as

(5.1)
$$\mathbb{C}$$

$$\downarrow$$

$$T_{n,m} \cong \mathbb{C}^k$$

$$\downarrow \psi^{n,m}_{-,n-1}$$

$$\mathbb{C} \xrightarrow{\psi^{2n-3,m-1}_{+,n-1}} B_{n-1,m-1} \xrightarrow{\psi^{n-1,m-1}_{+,n-2}} T_{n-2,m} \cong \mathbb{C}^k$$

We consider the following two cases.

Case 1. $\psi^{2n-3,m-1}_{+,n-1}$ is trivial. Then from the exactness of the horizontal sequence in (5.1), we know that $B_{n-1,m-1} \cong \mathbb{C}^{k-1}$ and $\psi^{n-1,m-1}_{+,n-2}$ is injective. Also, we conclude from the exactness of the vertical sequence in (5.1) that $\psi^{n,m}_{-,n-1}$ is surjective. However, from Proposition 5.5 we know that

$$\psi^{n-1,m-1}_{+,n-2} \circ \psi^{n,m}_{-,n-1} = 0.$$

Hence the only possibility is that $k = 1$, and this concludes that $T_{n,m} \cong \underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}$, and $B_{n,m-1} \cong B_{n-1,m-1} = 0$, which is the first case in the statement of the lemma.

Case 2. $\psi^{2n-3,m-1}_{+,n-1}$ is nontrivial. Then from the exactness of the horizontal sequence in (5.1), we know that $B_{n-1,m-1} \cong \mathbb{C}^{k+1}$ and $\psi^{n-1,m-1}_{+,n-2}$ is surjective. From the above discussion and the

bypass exact triangle from Proposition 3.10, we have another exact diagram

(5.2)
$$\underline{\mathrm{SHI}}(-Y(K), -\gamma_{(2,5-2n),m}) \cong \mathbb{C}$$

$$B_{n-1,m-1} \cong \mathbb{C}^{k+1} \xrightarrow{\psi_{+,n-2}^{n-1,m-1}} T_{n-2,m} \cong \mathbb{C}^k$$

$$\downarrow \psi_{-,n-3}^{n-2,m}$$

$$B_{n-3,m-1} \cong \mathbb{C}^{k+1}$$

The exactness of the vertical sequence in (5.2) implies that the map $\psi_{-,n-3}^{n-2,m}$ is injective. However, from Proposition 5.5, we have

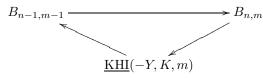$$\psi_{-,n-3}^{n-2,m} \circ \psi_{+,n-2}^{n-1,m-1} = 0.$$

Hence the only possibility is that $k = 0$. Thus, we conclude that $T_{n,m} \cong \underline{\mathrm{KHI}}(-Y, K, m) = 0$, which is the second case in in the statement of the lemma. $\square$

Lemma 5.8. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n_0 be a positive integer so that $Y_{-n_0}(K)$ is an instanton L-space. Suppose further that for a large enough integer n and some integer m with $|m| \leqslant g(K)$, we have $B_{n,m} = 0$. then one of the following two cases happens.*

(1) $\underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}$ and $T_{n,m} = 0$,
(2) $\underline{\mathrm{KHI}}(-Y, K, m) = 0$ and $B_{n,m-1} = 0$.

Proof. The proof is similar to that of Lemma 5.7. From Proposition 3.10, we have the following triangle

$$B_{n-1,m-1} \xrightarrow{\hspace{3cm}} B_{n,m}$$

$$\underline{\mathrm{KHI}}(-Y, K, m)$$

Hence there exists some $k \in \mathbb{N}$ so that

$$B_{n-1,m-1} \cong \underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}^k.$$

Also from Proposition 3.10, we have the following exact diagram

(5.3)
$$\mathbb{C}$$

$$\downarrow$$

$$T_{n,m} \qquad\qquad \mathbb{C}$$

$$\downarrow \psi_{-,n-1}^{n,m} \qquad\qquad \downarrow$$

$$\mathbb{C} \xrightarrow{\psi_{+,n-1}^{2n-3,m-1}} B_{n-1,m-1} \cong \mathbb{C}^k \xrightarrow{\psi_{+,n-2}^{n-1,m-1}} T_{n-2,m}$$

$$\downarrow \psi_{-,n-3}^{n-2,m-1}$$

$$B_{n-3,m-1} \cong \mathbb{C}^k$$

We consider the following two cases.

Case 1. $\psi_{+,n-1}^{2n-3,m-1}$ is trivial. Then from the exactness of the horizontal sequence in (5.3), we know that $T_{n-2,m} \cong \mathbb{C}^{k-1}$ and $\psi_{+,n-2}^{n-1,m-1}$ is surjective. Also, we conclude from the exactness of the second vertical sequence in (5.3) that $\psi_{-,n-3}^{n-2,m}$ is injective. However, from Proposition 5.5 we know that

$$\psi_{-,n-3}^{n-2,m} \circ \psi_{+,n-2}^{n-1,m-1} = 0.$$

Hence the only possibility is that $k = 1$. Hence we conclude that $\underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}$ and $T_{n,m} \cong T_{n-2,m} = 0$, which is the first case in the statement of the lemma.

Case 2. $\psi_{+,n-1}^{2n-3,m-1}$ is nontrivial. Then from the exactness of the horizontal sequence in (5.3), we know that $T_{n,m} \cong T_{n-2,m} \cong \mathbb{C}^{k+1}$ and $\psi_{+,n-2}^{n-1,m-1}$ is injective. Also, we conclude from the exactness of the first vertical sequence that $\psi_{-,n-1}^{n,m}$ is surjective. However, from Proposition 5.5 we know that

$$\psi_{+,n-2}^{n-1,m-1} \circ \psi_{-,n-1}^{n,m} = 0.$$

Hence the only possibility is that $k = 0$, and this concludes that

$$B_{n,m-1} \cong B_{n-1,m-1} \cong \underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}^k,$$

which is the second case in the statement of the lemma. $\square$

Proof of Theorem 5.1. By Definition 5.2 and Lemma 3.8, we know that

$$T_{n,g(K)+1} = 0 \text{ and } \underline{\mathrm{KHI}}(-Y, K, g(K) + 1) = 0.$$

We apply an induction that decreases the integer i: assuming that for $i + 1$, we have

$$\underline{\mathrm{KHI}}(-Y, K, i + 1) \cong \mathbb{C} \text{ or } 0$$

and either $T_{n,i+1} = 0$ or $B_{n,(i+1)-1} = 0$, then we want to prove the same results for i. When $T_{n,i+1} = 0$, from Lemma 5.7, we have either $\underline{\mathrm{KHI}}(-Y, K, i) \cong \mathbb{C}$ and $B_{n,i-1} = 0$ or $\underline{\mathrm{KHI}}(-Y, K, i) = 0$ and $T_{n,i} = 0$. When $B_{n,(i+1)-1} = 0$, from Lemma 5.8, we have either $\underline{\mathrm{KHI}}(-Y, K, i) \cong \mathbb{C}$ and $T_{n,i} = 0$ or $\underline{\mathrm{KHI}}(-Y, K, i) = 0$ and $B_{n,i-1} = 0$. Hence, the inductive step is completed and we conclude that

$$\underline{\mathrm{KHI}}(-Y, K, i) \cong \mathbb{C} \text{ or } 0.$$

for all $i \in \mathbb{Z}$ so that $|i| \leqslant g(K)$. From Lemma 3.8, we know that

$$\underline{\mathrm{KHI}}(-Y, K, i) \cong 0$$

for all $i \in \mathbb{Z}$ with $|i| > g(K)$. Hence we conclude the proof of Theorem 5.1. $\square$

5.2. **Coherent chains.**

In this subsection, we prove instanton analog of [RR17, Lemma 3.2] with more assumptions. First, we introduce the analog of [RR17, Definition 3.1] in instanton theory.

Definition 5.9. Suppose K is a knot in a rational homology sphere Y and suppose $\hat{\mu}$ is the meridian of K. Suppose the knot complement $Y(K)$ satisfying $H_1(Y(K)) \cong \mathbb{Z}$ so that we can identify $[\hat{\mu}] \in H_1(Y(K))$ as an integer q. Indeed, if a Seifert surface S of K is chosen, we can set $q = S \cdot \hat{\mu}$. For any integer s and its image $[s] \in \mathbb{Z}_q$, define

$$\underline{\mathrm{KHI}}(-Y, K, [s]) := \bigoplus_{k \in \mathbb{Z}} \underline{\mathrm{KHI}}(-Y, K, S, s + kq).$$

It is called a **positive chain** if it is generated by elements

$$x_1, \ldots, x_l, y_1, \ldots, y_{l-1},$$

each of which lives in a single grading associated to S and a single $\mathbb{Z}_2$-grading, and the differentials d_+ and d_- satisfy

$$d_-(y_i) \doteq x_{i+1}, d_+(y_i) \doteq x_i, \text{ and } d_-(x_i) = d_+(x_i) = 0 \text{ for all } i,$$

where $\doteq$ means equal up to multiplication by a unit. The space $\underline{\text{KHI}}(-Y, K, [s])$ is called a **negative chain** if there exist similar generators so that

$$d_-(x_i) \doteq y_i, d_+(x_i) \doteq y_{i-1}, \text{ and } d_-(y_i) = d_+(y_i) = 0 \text{ for all } i.$$

We call $\underline{\text{KHI}}(-Y, K)$ **consists of positive chains** if $\underline{\text{KHI}}(-Y, K, [s])$ is a positive chain for any $[s] \in \mathbb{Z}_q$ and **consists of negative chains** if $\underline{\text{KHI}}(-Y, K, [s])$ is a negative chain for any $[s] \in \mathbb{Z}_q$. We call $\underline{\text{KHI}}(-Y, K)$ **consists of coherent chains** if $\underline{\text{KHI}}(-Y, K)$ either consists of postive chains or consists of negative chains

Remark 5.10. By Definition 5.9, the space $\underline{\text{KHI}}(-Y, K, [s])$ is both a positive chain and a negative chain if and only if $\dim_{\mathbb{C}} \underline{\text{KHI}}(-Y, K, [s]) = 1$. By the proof of Proposition 3.33, the space $\underline{\text{KHI}}(-Y, K)$ consists of positive chains if and only if $\underline{\text{KHI}}(Y, K)$ consists of negative chains.

The main theorem in this subsection is the following.

Theorem 5.11. *Suppose $K \subset Y$ is a knot as in Definition 5.9. Note that $H_1(Y(K)) \cong \mathbb{Z}$. Suppose Y is an instanton L-space and suppose $n \in \mathbb{N}_+$. Suppose the basis $(\hat{\mu}, \hat{\lambda})$ of $\partial Y(K)$ is from Definition 3.5. If $Y_{-n}(K)$ is an instanton L-space, then $\underline{\text{KHI}}(-Y, K)$ consists of positive chains. If $Y_n(K)$ is an instanton L-space, then $\underline{\text{KHI}}(-Y, K)$ consists of negative chains.*

For simplicity, we only provide details of the proof for a special case of Theorem 5.11. The proof for the general case is similar. The main input is Theorem 5.1.

Definition 5.12. We adapt notations in Subsection 5.1 and Construction 3.21. For any integer s, suppose $B_{\geqslant s}^+$ is the subcomplex of B_s^+ with the underlying space

$$\bigoplus_{k \geqslant s} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq)$$

and suppose $B_{<s}^-$ is the subcomplex of B_s^- with the underlying space

$$\bigoplus_{k < s} \underline{\text{SHI}}(-Y(K), -\widehat{\Gamma}_\mu, S, s + kq).$$

Let $H(B_{\geqslant s}^+)$ and $H(B_{<s}^-)$ be the corresponding homologies.

Lemma 5.13. *For any integers n and i with $|i| \leqslant g(K)$, we have*

$$T_{n,i} \cong H(B_{\geqslant i}^+) \text{ and } B_{n,i} \cong H(B_{<i}^-).$$

Proof. This follows from Remark 5.3, equations (3.9) and (3.10), and Theorem 2.4. $\square$

Theorem 5.14. *Suppose K is a knot in an integral homology sphere Y with $\dim_{\mathbb{C}} I^\sharp(Y) = 1$. If there is a positive integer n so that $Y_{-n}(K)$ is an instanton L-space, then $\underline{\text{KHI}}(-Y, K)$ consists of positive chains in the sense of Definition 5.9.*

Proof. By Theorem 5.1, for any integer i, we have

$$\dim_{\mathbb{C}} \underline{\text{KHI}}(-Y, K, i) \leqslant 1.$$

Then we have integers

$$n_1 > n_2 > \cdots > n_k$$

so that

$$\dim_{\mathbb{C}} \underline{\text{KHI}}(-Y, K, i) = \begin{cases} 1 & \text{if } i = n_j \text{ for } j \in [0, k]; \\ 0 & \text{else.} \end{cases}$$

Suppose x_i is the generator of $\underline{\text{KHI}}(-Y, K, n_{2i-1})$ and y_i is the generator of $\underline{\text{KHI}}(-Y, K, n_{2i})$. We verify that those x_i and y_i satisfy the positive chain condition, *i.e.* for any integer i, we have

(5.4) $$d_-(y_i) \doteq x_{i+1}, d_+(y_i) \doteq x_i, \text{ and } d_-(x_i) = d_+(x_i) = 0,$$

where $\doteq$ means the equation holds up to multiplication by a unit. We prove this condition by induction. We only consider the condition about the differential d_+. The proof for d_- is similar. The gradings in the following arguments mean the gradings associated to the Seifert surface S. Note that by the proof of Theorem 5.1, we have

$$T_{n,n_{2l}} = B_{n,n_{2l-1}+1} = 0 \text{ for any } l.$$

Hence by Lemma 5.13, we have

$$T_{n,i} \cong H(B^+_{\geqslant n_{2l}}) = H(B^-_{<2l-1}).$$

First, suppose $i = 1$. Since x_1 lives in the top grading of $\underline{\text{KHI}}(-Y, K)$ and d_+ increases the $\mathbb{Z}$-grading, we must have $d_+(x_1) = 0$. Since $H(B^+_{\geqslant n_2}) = 0$ and there are only two generators x_1 and y_1 in $B^+_{\geqslant n_2}$, we must have $d_+(y_1) \doteq x_1$.

Then we assume the condition (5.4) holds for $i \leqslant l-1$ and prove it also holds for $i = l$. Since

$$H(B^+_{\geqslant n_{2l}}) = H(B^+_{\geqslant n_{2l-2}}) = 0,$$

we know the quotient complex $B^+_{\geqslant n_{2l}}/B^+_{\geqslant n_{2l-2}}$ also has trivial homology. Since it is generated by x_l and y_l, the coefficient of $d_+(y_l)$ about x_l must be nontrivial. Hence y_l is not in the $(n_{2l-1} - n_{2l} + 1)$-page of the spectral squence associated to d_+. Since other generators $x_1, \ldots, x_{l-1}, y_1, \ldots, y_{l-1}$ have smaller gradings than x_l, we know by construction of d_+ in Construction 2.6 that the coefficients of $d_+(y_l)$ about those generators are zeros. Hence $d_+(y_l) \doteq x_l$. Since $d_+ \circ d_+ = 0$, we have $d_+(x_l) = 0$. Thus, we prove the condition holds for $i = l$.

□

Proof of Theorem 5.11. If $Y_{-n}(K)$ is an instanton L-space, then the proof is similar to that of Theorem 5.14. To prove a generalization of Theorem 5.1, we need to remove the integral homology sphere assumption in Proposition 3.18 and Proposition 5.6. The corresponding proofs follow from Remark 3.19 and the proof of [BGW13, Proposition 4]. If $Y_n(K)$ is an instanton L-space, by Remark 5.10, we can consider the mirror knot to obtain the result.   □

5.3. **A graded version of Künneth formula.**

In this subsection, we prove the following graded version of Künneth formula for the connected sum of two knots.

Proposition 5.15. *Suppose Y_1 and Y_2 are two irreducible rational homology spheres and $K_1 \subset Y_1$, $K_2 \subset Y_2$ are two knots so that $Y_1(K_1)$ and $Y_2(K_2)$ are both irreducible. Suppose*

$$(Y', K') = (Y_1 \sharp Y_2, K_1 \sharp K_2)$$

is the connected sum of two knots. Then there is a minimal genus Seifert surface S of K' with the following properties.

(1) There is a 2-sphere $\Sigma \subset Y'$ intersecting the knot K' in two points and intersecting S in arcs.

(2) *If we cut S along $S \cap \Sigma^2$, then S decomposes into two surfaces $S_1 \subset Y_1$ and $S_2 \subset Y_2$ so that S_i is a union of some copies of Seifert surfaces of K_i for $i = 1, 2$.*

(3) *There is an isomorphism*

$$(5.5) \qquad \underline{\mathrm{KHI}}(Y', K', S, k) \cong \bigoplus_{i+j=k} \underline{\mathrm{KHI}}(Y_1, K_1, S_1, i) \otimes \underline{\mathrm{KHI}}(Y_2, K_2, S_2, j).$$

Proof. Let S be a minimal genus Seifert surface of K' and let $\Sigma \subset Y'$ be a 2-sphere so that Σ intersects K' in two points. We can choose Σ so that

$$\Sigma \cap \partial Y'(K') = \mu_1 \cup \mu_2,$$

where μ_1 and μ_2 are two meridians of K'. Write

$$A = \Sigma \cap Y'(K').$$

From now on, we also regard S as a surface inside the knot complement $Y'(K')$. We can isotope S so that S intersects A transversely and S has minimal intersections with both μ_1 and μ_2. Now we argue that we can further isotope S so that S intersects A in arcs. Suppose

$$S \cap A = \alpha_1 \cup \cdots \cup \alpha_n \cup \beta_1 \cup \cdots \cup \beta_m,$$

where α_i are arcs and β_j are closed curves. Observe that each component of $A \backslash (\alpha_1 \cup \cdots \cup \alpha_n)$ is a disk. Then using the arguments in the proof of [Rol90, Chapter 5, Theorem A14], we could further assume that $m = 0$, *i.e.*, S intersects A in arcs. When we cut the knot complement $Y'(K')$ along A, we obtain the disjoint union of the knot complements $Y_1(K_1)$ and $Y_2(K_2)$, and the surface S decomposes into $S_1 \subset Y_1(K_1)$ and $S_2 \subset Y_2$. Note that S_1 and S_2 must be the union of (possibly more than one) copies of Seifert surfaces of the corresponding knots. Then we prove the isomorphism (5.5).

First, we prove

$$(5.6) \qquad \underline{\mathrm{KHI}}(Y', K') \cong \underline{\mathrm{KHI}}(Y_1, K_1) \otimes \underline{\mathrm{KHI}}(Y_2, K_2).$$

To do so, we pick a meridian μ_i' of K_i for $i = 1, 2$ pick suitable orientations so that $(Y'(K'), \mu_1' \cup \mu_2')$ is a balanced sutured manifold. Then we can decompose it along the annulus A:

$$(Y'(K'), \mu_1' \cup \mu_2') \rightsquigarrow (Y_1(K_1), \mu_1 \cup \mu_1') \sqcup (Y_2(K_2), \mu_2 \cup \mu_2').$$

From [KM10b, Proposition 6.7], this annular decomposition leads to the isomorphism (5.6). To study the grading behavior of this isomorphism, we sketch the construction of the isomorphism as follows. Pick a connected oriented compact surface T so that

$$\partial T = -\mu_1 \cup -\mu_2.$$

Pick an annulus T' so that

$$\partial T' = -\mu_1' \cup -\mu_2'.$$

One could think of T' be a copy of the annulus A.

In [KM10b, Section 7], Kronheimer and Mrowka constructed closures of

$$(Y_1(K_1), \mu_1 \cup \mu_1') \sqcup (Y_2(K_2), \mu_2 \cup \mu_2')$$

as follows. First, glue $[-1, 1] \times (T \cup T')$ to $Y_1(K_1) \sqcup Y_2(K_2)$ using the boundary identifications as above to obtain a pre-closure

$$(5.7) \qquad \widetilde{M} = (Y_1(K_1) \sqcup Y_2(K_2)) \cup [-1, 1] \times (T \cup T').$$

The boundary of $\widetilde{M}$ has two components

$$\partial\widetilde{M} = R_+ \cup R_-,$$

where

$$R_\pm = R_\pm(\mu_1 \cup \mu_1') \cup R_\pm(\mu_2 \cup \mu_2') \cup \{\pm 1\} \times (T \cup T').$$

Second, choose an orientation preserving diffeomorphism

$$h : R_+ \to R_-$$

and use h to close up $\widetilde{M}$ and obtain a closed 3-manifold Y with a distinguishing surface R. The pair (Y, R) is a closure of $(Y_1(K_1), \mu_1 \cup \mu_1') \sqcup (Y_2(K_2), \mu_2 \cup \mu_2')$.

Remark 5.16. In [KM10b, Section 7], we also need to choose a simple closed curve in Y, either transversely intersecting R at one point or is non-separating on R, to achieve the irreducibility condition for related instanton moduli spaces. In the current proof, the choices of simple closed curves are straightforward, so we omit them from the discussion.

Note that gluing $[-1, 1] \times T_1$ to $(Y_1(K_1), \mu_1 \cup \mu_1') \sqcup (Y_2(K_2), \mu_2 \cup \mu_2')$ is the inverse operation of decomposing $(Y'(K'), \mu_1' \cup \mu_2')$ along the annulus A. As a result, (Y, R) is clearly a closure of $(Y'(K'), \mu_1' \cup \mu_2')$ as well. The identification of the closures induces the isomorphism in (5.6). More precisely, we can pick the surface T with large enough genus and pick a simple closed curve $\theta \subset T$ so that θ separates T into two parts, both of large enough genus, and with $-\mu_1'$ and $-\mu_2'$ sitting in different parts. We also pick a core θ' of the annulus T'. When choosing the gluing diffeomorphism $h : R_+ \to R_-$, we can choose one so that

$$(5.8) \qquad\qquad h(\{1\} \times \theta) = \{-1\} \times \theta, \text{ and } h(\{1\} \times \theta') = \{-1\} \times \theta'.$$

Hence, inside Y, there are two tori $S^1 \times \theta$ and $S^1 \times \theta'$. If we cut Y open along these two tori and reglue, then we obtain two connected 3-manifolds (Y_1, R_1) and (Y_2, R_2), which are closures of $(Y_1(K_1), \mu_1 \cup \mu_1')$ and $(Y_2(K_2), \mu_2 \cup \mu_2')$, respectively. The Floer's excision theorem in [KM10b, Section 7.3] then provide the desired isomorphism.

To study the gradings, recall that

$$S \cap A = \alpha_1 \cup \cdots \cup \alpha_n$$

where α_i are arcs connecting μ_1 to μ_2 on A. We can also regard those arcs as on the annulus T'. Assume that ∂S intersects each of μ_1' and μ_2' in n points as well. Note that we have assumed that T has a large enough genus. Then there are arcs $\delta_1,..., \delta_n$ so that the following holds. Recall we have chosen $\theta \subset T$ in previous above discussions.

(1) We have $\partial(\delta_1 \cup \cdots \cup \delta_n) = S \cap (\mu_1' \cup \mu_2')$.
(2) For $i = 1, .., n$, the arc δ_i intersects θ_1 transversely once.
(3) The surface $S \backslash (\delta_1 \cup \cdots \cup \delta_n \cup \theta_1)$ also has two components.
(4) Let $\widetilde{S} = S \cup [-1, 1] \times (\alpha_1 \cup \cdots \cup \alpha_n)$ be a properly embedded surface inside the pre-closure $\widetilde{M}$ as in (5.7), then we can choose a gluing diffeomorphism $h : R_+ \to R_-$ satisfying the condition (5.8) and the following extra condition

$$h(\partial\widetilde{S} \cap R_+) = \partial\widetilde{S} \cap R_-.$$

Hence, the surface S extends to a closed surface $\bar{S} \subset Y$ that induces the desired $\mathbb{Z}$-grading on $\underline{\mathrm{KHI}}(Y', K')$. When we cut Y open along $S^1 \times \theta$ and $S^1 \times \theta'$ and reglue, the surface $\bar{S}$ is also cut and reglued to form two closed surfaces $\bar{S}_1 \subset Y_1$ and $\bar{S}_2 \subset Y_2$. They are the extensions of the Seifert

surface S_1 of K_1 and the Seifert surface S_2 of K_2 in the corresponding closures. Hence the Floer's excision theorem in [KM10b, Section 7.3] provides desired the isomorphism (5.5). □

5.4. **Proofs of theorems in the introduction.**

In this subsection, we prove Theorem 1.9, Theorem 1.11, and Theorem 1.17.

Proof of Theorem 1.9. By Remark 1.8, we may assume $S_n^3(K)$ is an instanton L-space for some $n \in \mathbb{N}_+$. Then by Theorem 5.11, the space $\underline{\mathrm{KHI}}(S^3, K)$ consists of coherent chains. Then arguments about $KHI(S^3, K, S, i)$ follow from Definition 5.9 and Proposition 3.37.

To prove K is a prime knot, we can apply the proof of [BVV18, Corollary 1.4] to KHI, replacing [BVV18, Theorem 1.1] by [BS18a, Theorem 1.7]. Note that we need the graded version of Künneth formula for KHI in Proposition 5.15. □

Proof of Theorem 1.17. By (1.3), a knot $K \subset Y$ is instanton Floer simple if and only if the mirror knot $(-Y, K)$ is instanton Floer simple. Note that by spectral sequences in Theorem 3.20, we always have

$$\dim_{\mathbb{C}} \underline{\mathrm{KHI}}(-Y, K, [s]) \geqslant \dim_{\mathbb{C}} I^\sharp(-Y, [s]) \geqslant 1.$$

By Remark 5.10, we know that $(-Y, K)$ is instanton Floer simple if and only if the space $\underline{\mathrm{KHI}}(-Y, K)$ both consists of positive chains and consists of negative chains.

By Theorem 3.25 and Theorem 3.32, if K is instanton Floer simple, then for any large integer n, the manifolds $Y_n(K)$ and $Y_{-n}(K)$ are both instanton L-spaces. By the similar argument in the proof of [BGW13, Proposition 4], the manifold $Y_r(K)$ is an instanton L-space for any $|r| \geqslant n$.

Conversely, if for any r with $|r|$ sufficiently large, the manifold $Y_r(K)$ is an instanton L-space, then for any large integer n, the manifolds $Y_n(K)$ and $Y_{-n}(K)$ are both instanton L-spaces. By Proposition 5.11, the space $\underline{\mathrm{KHI}}(-Y, K)$ both consists of positive chains and consists of negative chains. Hence K is an instanton Floer simple knot. □

Finally, we prove Theorem 1.11. Suppose $K \subset Y$ is a knot with $H_1(Y(K)) \cong \mathbb{Z}$ and suppose $\hat{\mu}$ is the meridian of K with $q = \in = S \cdot \hat{\mu}$, where S is the Seifert surface of K. We choose a basis $(\hat{\mu}, \hat{\lambda})$ of $H_1(\partial Y(K))$ as in Definition 3.5 and identify the slope with rational numbers. Then we have the following lemma.

Lemma 5.17 ([RR17, Lemma 2.7]). *Consider the setting as above. If $r = u/v$, the manifold $Y_r(K)$ is obtained from $Y' = Y \sharp L(v, -u)$ by some integral surgery on $K' = K \sharp K(v, -u, 1)$, where $K(v, -u, 1)$ is the unique knot in $L(v, -u)$ so that the complement is diffeomorphic to $S^1 \times D^2$. Moreover, we have*

$$H_1(Y'(K')) \cong H_1(Y(K)) \cong H_1(S^1 \times D^2)/(\hat{\mu}, \mu'),$$

where μ' is the meridian of $K(v, -u, 1)$. Hence $H_1(Y'(K')) \cong \mathbb{Z}$ if and only if $\gcd(q, v) = 1$.

Proof of Theorem 1.11. By [RR17, Lemma 3.2], for a Heegaard Floer L-space knot $K \subset Y$, the space $\widehat{HFK}(Y, K)$ satisfies similar coherent chain condition as in Definition 5.9. Consider the $\mathbb{Z}$-grading on $\widehat{HFK}(Y, K)$ induced by pairing the first Chern class of the spin^c structure with S. Since $H_1(Y(K)) \cong \mathbb{Z}$, the $\mathbb{Z}$-grading encodes all information in the spin^c decomposition and the coherent chain condition implies

$$\dim_{\mathbb{F}_2} \widehat{HFK}(Y, K, S, i) \leqslant 1.$$

Hence the dimension is determined by the graded Euler characteristic.

If $v = 1$ and $r \in \mathbb{Z}$, then by similar discussion as above, Theorem 5.11 implies that $\underline{\mathrm{KHI}}(Y, K, S, i)$ is determined by the graded Euler characteristic. Hence the theorem follows from (1.4).

If $v \neq 1$, then by Lemma 5.17, we can apply the proof for $v = 1$ to

$$(Y', K') = (Y \sharp L(v, -u), K \sharp K(v, -u, 1)).$$

Note that simple knots are instanton Floer simple knots by [LY20, Proposition 1.7]. Then the theorem follows from the graded Künneth formula for $\underline{\mathrm{KHI}}$ (Proposition 5.15) and $\widehat{HFK}$ ([OS11, Section 5]). We do not need to consider the irreducible condition due to the convention in Subsection 3.2. □

6. Dehn surgeries along genus-one knots

In this section, we study the framed instanton Floer homology of Dehn surgeries along knots that satisfies the following conditions:

(1) The genus of the knot is 1, i.e., $g(K) = 1$.
(2) The instanton knot homology of the knot is determined by the Alexander polynomial, i.e.,

$$\Delta_K(t) = a_1 t + a_0 + a_{-1} \text{ and } \dim_{\mathbb{C}} \underline{\mathrm{KHI}}(S^3, K, i) = |a_i| \text{ for } i \in \mathbb{Z}.$$

Such knots include all genus-one Khovanov-thin knots (in particular, genus-one quasi-alternating knots [KM11, Corollary 1.6]). In Table 1, we list all genus-one alternating knots with crossings $\leqslant 12$ (they are also all known examples of genus-one quasi-alternating knots). The data are from KnotInfo [LM21]. Note that we normalize the Alexander polynomial by (1.1). The first knot for each crossing number in the table is a twisted knot. The reader can compare this table with examples in [BS20a].

TABLE 1. genus-one alternating knots with crossings $\leqslant 12$

No.	Name	4-ball genus	Signature	Two-bridge notation	Alexander polynomial
1	3_1	1	-2	3/1	$t - 1 + t^{-1}$
2	4_1	1	0	5/2	$-t + 3 - t^{-1}$
3	5_2	1	-2	7/3	$2t - 3 + 2t^{-1}$
4	6_1	0	0	9/7	$-2t + 5 - 2t^{-1}$
5	7_2	1	-2	11/5	$3t - 5 + 3t^{-1}$
6	7_4	1	-2	15/11	$4t - 7 + 4t^{-1}$
7	8_1	1	0	13/11	$-3t + 7 - 3t^{-1}$
8	8_3	1	0	17/4	$-4t + 9 - 4t^{-1}$
9	9_2	1	-2	15/7	$4t - 7 + 4t^{-1}$
10	9_5	1	-2	23/17	$6t - 11 + 6t^{-1}$
11	9_{35}	1	-2		$7t - 13 + 7t^{-1}$
12	10_1	1	0	17/15	$-4t + 9 - 4t^{-1}$
13	10_3	0	0	25/6	$-6t + 13 - 6t^{-1}$
14	$11a_{247}$	1	-2	19/17	$5t - 9 + 5t^{-1}$
15	$11a_{343}$	1	-2	31/27	$8t - 15 + 8t^{-1}$
16	$11a_{362}$	1	-2		$10t - 19 + 10t^{-1}$
17	$11a_{363}$	1	-2	35/29	$9t - 17 + 9t^{-1}$
18	$12a_{803}$	1	0	21/2	$-5t + 11 - 5t^{-1}$
19	$12a_{1166}$	1	0	33/4	$-8t + 17 - 8t^{-1}$
20	$12a_{1287}$	1	0	37/6	$-9t + 19 - 9t^{-1}$

From conditions in (1.1), there are two possibilities of the Alexander polynomial:

(i) $\Delta_K(t) = at - (2a - 1) + at^{-1}$ for some $a \in \mathbb{N}_+$;

(ii) $\Delta_K(t) = -at + (2a + 1) - at^{-1}$ for some $a \in \mathbb{N}_+$.

We treat these two cases separately in the following two subsections.

Convention. For simplicity, we write $\underline{\mathrm{KHI}}(K)$ for $\underline{\mathrm{KHI}}(-S^3, K)$ and $\underline{\mathrm{KHI}}(K, i)$ for $\underline{\mathrm{KHI}}(-S^3, K, S, i)$, where S is a Seifert surface of K. Recall that we write $\bar{K}$ for the mirror knot of K. We will write $H(C)$ for the homology of a complex C and write f_* for the induced map between homologies.

Recall the results from Section 3.5. In this case, we have $A_s = (\underline{\mathrm{KHI}}(K), d_s)$ for any s, and

$$d_s(x) = \begin{cases} d_+(x) & \mathrm{gr}(x) > s, \\ d_+(x) + d_-(x) & \mathrm{gr}(x) = s, \\ d_-(x) & \mathrm{gr}(x) < s. \end{cases}$$

where $\mathrm{gr}(x)$ is the grading of $x \in \underline{\mathrm{KHI}}(K)$ associated to the Seifert surface. We can further decompose the differentials as follows:

$$d_+ = \sum_{i<j} d_j^i \text{ and } d_- = \sum_{i>j} d_j^i, \text{ where } d_j^i : \underline{\mathrm{KHI}}(K, i) \to \underline{\mathrm{KHI}}(K, j).$$

Since $g(K) = 1$, the -3-surgery is a large surgery in the sense of Theorem 1.22. Hence we have

$$I^\sharp(-S^3_{-3}(K)) \cong \bigoplus_{s=-1}^{1} H(A_s, d_s),$$

where

$$H(A_1, d_1) \cong H(\underline{\mathrm{KHI}}(K), d_-) \cong I^\sharp(-S^3) \cong \mathbb{C},$$

and

$$H(A_{-1}, d_{-1}) \cong H(\underline{\mathrm{KHI}}(K), d_+) \cong I^\sharp(-S^3) \cong \mathbb{C}.$$

Hence we know that

(6.1) $$\dim_{\mathbb{C}} I^\sharp(-S^3_{-3}(K)) = 2 + \dim_{\mathbb{C}} H(A_0, d_0).$$

Since $a_1 = a_{-1}$, by the graded Euler characteristic of KHI [Lim09, KM10a], we know that that the parities of $\underline{\mathrm{KHI}}(K, 1)$ and $\underline{\mathrm{KHI}}(K, -1)$ are the same under the $\mathbb{Z}_2$ grading. By Proposition 3.37, we know that there is no d^1_{-1} or d^{-1}_1 differentials. Hence, we know that

$$d_0 = d_1^0 + d_{-1}^0.$$

6.1. The case of $(2a + 1)$.

In this case we know that

$$\underline{\mathrm{KHI}}(K, i) \cong \begin{cases} \mathbb{C}^a & i = \pm 1, \\ \mathbb{C}^{2a+1} & i = 0, \\ 0 & \text{else.} \end{cases}$$

We have the following.

Lemma 6.1. *The differential*

$$d_0^1 : \underline{\mathrm{KHI}}(K, 1) \to \underline{\mathrm{KHI}}(K, 0)$$

is injective and the differential

$$d_{-1}^0 : \underline{\mathrm{KHI}}(K, 0) \to \underline{\mathrm{KHI}}(K, -1)$$

is surjective.

Proof. Since
$$\dim_{\mathbb{C}} \operatorname{Ker}(d_{-1}^0) \geqslant \dim_{\mathbb{C}} \underline{\operatorname{KHI}}(K, 0) - \dim_{\mathbb{C}} \underline{\operatorname{KHI}}(K, -1) = a + 1$$
and
$$\dim_{\mathbb{C}} \operatorname{Im}(d_0^1) \leqslant \dim_{\mathbb{C}} \underline{\operatorname{KHI}}(K, 1) = a,$$
we know
$$1 \leqslant \dim_{\mathbb{C}}(\operatorname{Ker}(d_{-1}^0)/\operatorname{Im}(d_0^1)) \leqslant \dim_{\mathbb{C}} H(A_1, d_1) = 1.$$
We conclude that
$$\dim_{\mathbb{C}} \operatorname{Ker}(d_{-1}^0) = a + 1$$
which means that d_{-1}^0 is surjective. Also, we must have
$$\dim_{\mathbb{C}} \operatorname{Im}(d_0^1) = a + 1$$
which means that d_0^1 is injective. $\qquad\square$

Lemma 6.2. *We have* $\operatorname{Ker}(d_1^0) = \operatorname{Ker}(d_{-1}^0) \cong \mathbb{C}^{a+1}$.

Proof. Applying the argument in Lemma 6.1 to the bent complex (A_{-1}, d_{-1}), we also conclude that
$$\dim_{\mathbb{C}} \operatorname{Ker}(d_{-1}^0) = a + 1.$$
Hence $\operatorname{Ker}(d_1^0) \cong \operatorname{Ker}(d_{-1}^0)$. Then we show they are indeed the same space. Suppose
$$x \in \operatorname{Ker}(d_{-1}^0) \text{ so that } x \notin \operatorname{Ker}(d_1^0).$$
Then we know that
$$d_0^1 \circ d_1^0(x) \neq 0.$$
Since $x \in \operatorname{Ker}(d_{-1}^0)$ and $\operatorname{Im}(d_0^1) \subset \operatorname{Ker}(d_{-1}^0)$, the map
$$(d_0^1 \circ d_1^0)_* : H(\underline{\operatorname{KHI}}(K, 0) \xrightarrow{d_{-1}^0} \underline{\operatorname{KHI}}(K, -1)) \to H(\underline{\operatorname{KHI}}(K, 0) \xrightarrow{d_{-1}^0} \underline{\operatorname{KHI}}(K, -1))$$
is non-trivial. By Lemma 5.13, we can identify the map $(d_0^1 \circ d_1^0)_*$ between bent complexes with the composition of bypass maps
$$\psi_{-,n}^{n+1} \circ \psi_{+,n+1}^{n+2} = 0 : B_{n+2,i} \to B_{n,i}.$$
By Proposition 5.5, this map is zero, which is a contradiction. Hence, we conclude that
$$\operatorname{Ker}(d_{-1}^0) \subset \operatorname{Ker}(d_1^0).$$
Since they have the same dimension, they must be the same vector space. $\qquad\square$

Proposition 6.3. *Suppose K is a genus-one knot so that*
$$\Delta_K(t) = at + (2a + 1) + at^{-1} \text{ for } a \in \mathbb{N}_+ \text{ and } \dim_{\mathbb{C}} \underline{\operatorname{KHI}}(K) = 4a + 1.$$
Then for any $u, v \in \mathbb{Z}$ with $u \neq 0, v > 0$ and $\gcd(u, v) = 1$, we have
$$\dim_{\mathbb{C}} I^\sharp(S_{u/v}^3(K)) = 2av + |u|.$$

Proof. Applying Lemma 6.2 to K, we have
$$\dim_{\mathbb{C}} H(A_0, d_0) = 2a + 1.$$
By (6.1), we conclude that
$$\dim_{\mathbb{C}} I^\sharp(-S_{-3}^3(K)) = 2 + \dim_{\mathbb{C}} H(A_0, d_0) = 2a + 3.$$

The same argument applies to the mirror $\bar{K}$ of K, so we know that

$$\dim_{\mathbb{C}} I^{\sharp}(-S_3^3(K)) = \dim_{\mathbb{C}} I^{\sharp}(-S_{-3}^3(\bar{K})) = 2a + 3.$$

Then the proposition follows from [BS20a, Theorem 1.1]. $\qquad\square$

Remark 6.4. Under the terminologies in [BS20a], we know that $r_0(K) = 2a$ and $\nu^{\sharp}(K) = 0$ under the assumption of Proposition 6.3. However, we do not know if K is V-shaped or W-shaped in the sense of [BS20a, Definition 3.6]. If K is slice, then [BS20a, Theorem 3.7] implies it is W-shaped. If we knew the shape, then [BS20a, Theorem 1.1] would also tell us $\dim_{\mathbb{C}} I^{\sharp}(S_0^3(K))$.

6.2. The case of $(2a - 1)$.

In this case we know that

$$\underline{\mathrm{KHI}}(K, i) \cong \begin{cases} \mathbb{C}^a & i = \pm 1, \\ \mathbb{C}^{2a-1} & i = 0, \\ 0 & \text{else.} \end{cases}$$

Since

$$\mathrm{Ker}(d_0^1) \subset H(A_1, d_1) \cong \mathbb{C}$$

hence we must have

$$\dim_{\mathbb{C}} \mathrm{Ker}(d_0^1) \leqslant 1.$$

Hence we have the following two subcases.

(1) $\dim_{\mathbb{C}} \mathrm{Ker}(d_0^1) = 0$.
(2) $\dim_{\mathbb{C}} \mathrm{Ker}(d_0^1) = 1$.

Lemma 6.5. *We have* $\mathrm{Ker}(d_1^0) = \mathrm{Ker}(d_{-1}^0) \cong \mathbb{C}^a$ *in Case (1).*

Proof. The condition $\dim_{\mathbb{C}} \mathrm{Ker}(d_0^1) = 0$ implies d_0^1 is injective and $\dim_{\mathbb{C}} \mathrm{Im}(d_0^1) = a$. Since $\mathrm{Im}(d_0^1) \subset \mathrm{Ker}(d_{-1}^0)$, we know that $\dim_{\mathbb{C}} \mathrm{Ker}(d_{-1}^0) \geqslant a$ and hence $\dim_{\mathbb{C}} \mathrm{Im}(d_{-1}^0) \leqslant a - 1$. Since

$$\underline{\mathrm{KHI}}(K, -1)/(\mathrm{Im}(d_{-1}^0)) \subset H(A_1, d_1) \cong \mathbb{C},$$

we must have $\dim_{\mathbb{C}} \mathrm{Ker}(d_{-1}^0) = a$.

Since d_0^1 is injective, by the proof of Lemma 6.2, we know that $\mathrm{Ker}(d_{-1}^0) \subset \mathrm{Ker}(d_1^0)$. Hence we know that $\dim_{\mathbb{C}} \mathrm{Ker}(d_{-1}^0) \geqslant a$ and hence $\dim_{\mathbb{C}} \mathrm{Im}(d_{-1}^0) \leqslant a - 1$. Since

$$\underline{\mathrm{KHI}}(K, 1)/(\mathrm{Im}(d_1^0)) \subset H(A_{-1}, d_{-1}) \cong \mathbb{C},$$

we must have $\dim_{\mathbb{C}} \mathrm{Ker}(d_1^0) = a = \dim_{\mathbb{C}} \mathrm{Ker}(d_{-1}^0)$ and hence $\mathrm{Ker}(d_1^0) = \mathrm{Ker}(d_{-1}^0)$. $\qquad\square$

To distinguish the bent complexes of K and its mirror $\bar{K}$, we write $A_s(K)$ and $A_s(\bar{K})$, respectively. We write $\bar{d}_j^i$ for the component of differentials in $A_s(\bar{K})$.

Lemma 6.6. *We have* $\mathrm{Ker}(d_1^0) = \mathrm{Ker}(d_{-1}^0) \cong \mathbb{C}^{a-1}$ *in Case (2).*

Proof. Note that $\mathrm{Ker}(d_0^1) \subset H(A_1, d_1) \cong \mathbb{C}$. This means that

$$\mathrm{Ker}(d_{-1}^0) = \mathrm{Im}(d_0^1) \text{ and } \mathrm{Im}(d_{-1}^0) = \underline{\mathrm{KHI}}(K, -1).$$

Consider the bent complex of the mirror knot. By Proposition 3.33 and Corollary 3.35, we have a duality between d_j^i and $\bar{d}_i^j$. In particular, we have

$$\mathrm{Ker}(\bar{d}_0^{-1}) \cong \mathrm{Coker}(d_{-1}^0) = 0.$$

So we can apply Lemma 6.5 to $A_s(\bar{K})$ and conclude that $\mathrm{Ker}(\bar{d}_1^0) = \mathrm{Ker}(\bar{d}_{-1}^0)$. Using the duality again, we have $\mathrm{Im}(d_0^1) = \mathrm{Im}(d_0^{-1}) \cong \mathbb{C}^{a-1}$. Hence $\mathrm{Ker}(d_0^{-1}) \cong \mathbb{C}$. Since

$$\mathrm{Ker}(d_0^{-1}) \subset H(A_{-1}, d_{-1}),$$

we conclude that

$$\mathrm{Ker}(d_{-1}^0) = \mathrm{Im}(d_0^1) = \mathrm{Im}(d_0^{-1}) = \mathrm{Ker}(d_1^0).$$

$\square$

The following corollary is straightforward from the above discussion.

Corollary 6.7. *For a knot $K \subset S^3$, its bent complex $A_s(K)$ falls into Case (1) if and only if $A_S(\bar{K})$ falls into Case (1).*

Proposition 6.8. *Suppose K is a genus-one knot so that*

$$\Delta_K(t) = at + (2a - 1) + at^{-1} \text{ for } a \in \mathbb{N}_+ \text{ and } \dim_{\mathbb{C}} \underline{\mathrm{KHI}}(K) = 4a - 1.$$

Then for any $u, v \in \mathbb{Z}$ with $u \neq 0, v > 0$ and $\gcd(u, v) = 1$, one and exactly one of the following two cases happens.
(a) $\dim_{\mathbb{C}} I^{\sharp}(S^3_{u/v}(K)) = (2a - 1)v + |u - v|$.
(b) $\dim_{\mathbb{C}} I^{\sharp}(S^3_{u/v}(K)) = (2a - 1)v + |u + v|$.

Proof. When $A_s(K)$ falls into Case (1), then from Lemma 6.5 we know that

$$\dim_{\mathbb{C}} H(A_0, d_0) = 2a + 1.$$

Hence by (6.1), we conclude that

$$\dim_{\mathbb{C}} I^{\sharp}(-S^3_{-3}(K)) = 2 + \dim_{\mathbb{C}} H(A_0, d_0) = 2a + 3.$$

Furthermore, by Corollary 6.7, we know that $A_s(\bar{K})$ falls into Case (1). By Lemma 6.6, it follows that

$$\dim_{\mathbb{C}} I^{\sharp}(-S^3_3(K)) = \dim_{\mathbb{C}} I^{\sharp}(-S^3_{-3}(\bar{K})) = 2a + 1.$$

Then from [BS20a, Theorem 1.1] we know that Case (a) holds. When $A_s(K)$ of K falls into Case (1), by similar proof, we know that Case (b) holds. $\square$

Remark 6.9. Note that K satisfies Case (a) in Proposition 6.8, if and only if $\bar{K}$ satisfies Case (b) in Proposition 6.8. The hypothesis of Proposition 6.8 only involves the genus, the Alexander polynomial, and the total dimension of the instanton knot homology of the knot, which are all impossible to distinguish K from its mirror.

Remark 6.10. The two cases of Proposition 6.3 correspond to the two cases where $\nu^{\sharp}(K) = 1$ and $\nu^{\sharp}(K) = -1$, respectively. For genus-one alternating knots, from [BS20a, Corollary 1.10] we know that

$$\tau^{\sharp}(K) = -\frac{1}{2}\sigma(K), |\sigma(K)| \leqslant 2,$$

$$2\tau^{\sharp}(K) - 1 \leqslant \nu^{\sharp}(K) \leqslant 2\tau^{\sharp}(K) + 1,$$

and hence

$$-1 \leqslant \nu^{\sharp}(K) \leqslant 1.$$

If we suppose futher that the Alexander polynomial is of the form

$$\Delta_K(t) = at + (2a - 1) + a^{-1},$$

then we have $\sigma(K) \neq 0$ and hence $\tau^\sharp(K) = \nu^\sharp(K) = -\sigma(K)/2$. Thus, for genus-one alternating knots, which case of Proposition 6.8 happens depends on the signature of K.

Proof of Theorem 1.20. The result in instanton theory is a combination of Proposition 6.3, Proposition 6.8, Remark 6.4, and Remark 6.10. The result in Heegaard Floer theory is follows from [Han20, Proposition 15]. □

7. Examples of SU(2)-abundant knots

In this subsection, we provide many examples of $SU(2)$-abundant knots.

Proposition 7.1. *Instanton L-space knots in S^3 are classified in the following cases*

(1) An alternating knot is an instanton L-space knot if and only if it is the torus knots $T(2, 2n+1)$.
(2) A Montesinos knots (in particular, a pretzel knot) is an instanton L-space knot if and only if it is the torus knot $T(2, 2n+1)$, the pretzel knot $P(-2, 3, 2n+1)$ for $n \in \mathbb{N}_+$, and their mirrors.
(3) Knots that are closures of 3-braids are not instanton L-space knots except the twisted torus knots $K(3, q; 2, p)$ with $pq > 0$ and their mirrors.

Proof. Note that torus knots admit lens spaces surgeries [Mos71] and pretzel knots $P(-2, 3, 2n+1)$ admit Seifert fibred L-space surgeries [LM16]. Hence they are instanton L-space knots.

Theorem 1.3 provides many necessary conditions of instanton L-space knots. By [OS05b, Proposition 4.1], if an alternating knot satisfies term (1) in Theorem 1.3, then it is the $T(2, 2n+1)$ torus knot. Hence hyperbolic alternating knots are not instanton L-space knots.

In [BM18], there is a classification of (Heegaard Floer) L-space knots for Montesinos knots. From [BM18, Section 3.1], the proof of this classification only depends on term (1) in Theorem 1.3, the inequality (1.2), the fibredness, and the strongly quasi-positive condition [BS19, Theorem 1.5]. Hence the classification also works for instanton L-space knots.

In [LV21], it is shown that all closures of 3-braid except $K(3, q; 2, p)$ do not satisfy term (1) and term (2) in Theorem 1.3. Hence they are not instanton L-space knots. □

Remark 7.2. For pretzel knots, there is another approach [LM16] to classify L-space knots, which only depends on term (1) in Theorem 1.3, the inequality (1.2), the fibredness, and the direct calculation on $\widehat{HFK}(S^3, P(3, -5, 3, -2))$. However, it is hard to calculate $KHI(S^3, P(3, -5, 3, -2))$ directly, so we use the approach in [BM18].

Remark 7.3. Note that $K = K(3, q; 2, p)$ with $pq > 0$ is a $(1, 1)$-L-space knot from the proof of [Vaf15, Theorem 3.1(a)]. By [LY20, Corollary 1.5], we know that $\dim_\mathbb{C} KHI(S^3, K) = \dim_{\mathbb{F}_2} \widehat{HFK}(S^3, K)$. However, we do not know if K is an instanton L-space knot because [Vaf15, Theorem 3.1(a)] depends on the calculation of the chain complex $CFK^-(S^3, K)$ by a genus one doubly-pointed Heegaard diagram.

Proof of Corollary 1.6. This follows directly from Proposition 7.1 and Remark 1.8. □

Remark 7.4. There is a family of twisted torus knots $K(p, q; 2, m)$ with some conditions in [Mor06, Theorem 5] whose Alexander polynomials do not satisfy term (1) in Theorem 1.3. Thus, those knots are also not instanton L-space knots and hence $SU(2)$-abundant. In general, the classification of L-space knots for twisted torus knots is still open; see [Vaf15, Mot16, BM19] for some special cases.

Then we consider satellite knots and cable knots. There are some useful theorems.

Definition 7.5 ([SZ20]). A knot $K \subset S^3$ is called $SU(2)$-**averse** if there are infinitely many $r \in \mathbb{Q}\{0\}$ so that all representations $\pi_1(S^3_r(K)) \to SU(2)$ have abelian images.

Remark 7.6. If $b_1(Y) = 0$, then an $SU(2)$ representation of Y has abelian image if and only if it has cyclic image.

Theorem 7.7 ([SZ20, Theorem 1.8]). *Let $K \subset S^3$ be a nontrivial knot, and suppose that some satellite $P(K)$ with winding number w is $SU(2)$-averse. Then we have the following.*

(1) If $P(U)$ is not the unknot U, then it is also $SU(2)$-averse.
(2) If $w = 0$, then K is $SU(2)$-averse.

Theorem 7.8 ([SZ20, Theorem 10.6]). *Let $K \subset S^3$ be a nontrivial knot, and let $p, q \in \mathbb{Z}$ satisfying $\gcd(p, q) = 1$ and $q \geqslant 2$. If cable knot $K_{p,q}$ of K is $SU(2)$-averse, then K is also $SU(2)$-averse.*

Theorem 7.9 ([BS19, Lemma 8.5]). *Let $K \subset S^3$ be a nontrivial knot, and let $p, q \in \mathbb{Z}$ satisfying $\gcd(p, q) = 1, q \geqslant 2$, and $p/q > 2g(K) - 1$. Then the cable knot $K_{p,q}$ is a positive instanton L-space knot if and only if K is an instanton L-space knot.*

Definition 7.10. A $K \subset S^3$ is called a **distinguished knot** if it is an alternating knot, a Montesinos knot, or a knot from a 3-braid except the unknot, $T(2, 2n+1)$, $P(-2, 3, 2n+1)$ with $n \in \mathbb{N}_+$, $K(3, q; 2, p)$ with $pq > 0$, and their mirrors.

Note that distinguished knots are not instanton L-space knots and hence not $SU(2)$-averse by Remark 1.8. Then we have the following corollaries.

Corollary 7.11. *Suppose $P(K) \subset S^3$ is a satellite knot with winding number $w \geqslant 0$ of the pattern $P \subset S^1 \times D^2$. If one of the following holds, then $P(K)$ is not $SU(2)$-averse:*

(1) $P(U)$ is a distinguished knot;
(2) $w \neq 0$ and K is a distinguished knot.

Corollary 7.12. *Let $K \subset S^3$ be a distinguished knot, and let $p, q \in \mathbb{Z}$ satisfying $\gcd(p, q) = 1, q \geqslant 2$, and $p/q > 2g(K) - 1$. Then the cable knot $K_{p,q}$ is $SU(2)$-abundant.*

Finally, we strengthen a result in [BS19, Theorem 1.8].

Corollary 7.13. *Suppose $K \subset S^3$ is a nontrivial knot and suppose $S^3_3(K)$ does not have irreducible $SU(2)$ representations. Then K is a prime, fibred, strongly quasi-positive knot of genus two, and its instanton knot homology has the form*

$$(7.1) \qquad\qquad \dim_{\mathbb{C}} KHI(S^3, K, S, i) = \begin{cases} 1 & |i| \leqslant 2, \\ 0 & \text{else.} \end{cases}$$

Proof. By Remark 1.8, we know that K is an instanton L-space. Then Theorem 1.9 applies. By [BS19, Theorem 1.8] we know K is fibred, so [BS19, Theorem 1.11] applies and we obtain (7.1). □

Remark 7.14. In [LL21], the first author and Liang proved that if $KHI(S^3, K)$ has the form (7.1) for some knot $K \subset S^3$, then K must be an instanton L-space knot. Then by [BS19, Theorem 1.5], we know that $S^3_3(K)$ must be an instanton L-space. However, it is not enough to figure out whether $S^3_3(K)$ has irreducible $SU(2)$ representations.

8. Further directions

In this section, we discuss some further directions of techniques introduced in this paper.

First, in Heegaard Floer homology, Ozsváth and Szabó [OS08, OS11] introduced a mapping cone formula. Roughly speaking, for a null-homologous knot K in a closed 3-manifold Y, the homology $\widehat{HF}(Y_r(K))$ for any slope r can be computed by the filtrations on $\widehat{CF}(Y)$ induced by K and $-K$. The large surgery formula is the first step of their proof, which is recovered in instanton theory by Theorem 1.22. To prove an analog of the mapping cone formula in instanton theory. We need to further recover the following structures.
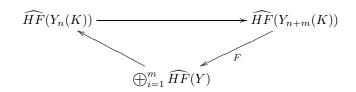
Fact. Suppose K is a null-homologous in a closed 3-manifold Y. For any integer n, suppose $W_n(K)$ is the cobordism from Y to $Y_n(K)$ induced by attaching 4-dimensional 2-handle and suppose $W'_n(K)$ is the cobordism from $Y_n(K)$ to Y obtained from $-W_n(K)$ by turning around two ends. We have the following structures in Heegaard Floer theory.

(1) There is a spin^c decomposition of the cobordism map:

$$\widehat{HF}(W_n(K)) = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(W_n(K))} \widehat{HF}(W_n(K),\mathfrak{s}) : \widehat{HF}(Y) \to \widehat{HF}(Y_n(K)).$$

Also, there is a spin^c decomposition of $\widehat{HF}(W'_n(K))$.

(2) For a large enough n, the spin^c decomposition of $\widehat{HF}(W'_n(K))$ is compatible with some maps constructed by the filtrations on $\widehat{CF}(Y)$ from K and $-K$.

(3) For any integer n and any positive integer m, there is a generalized surgery exact triangle
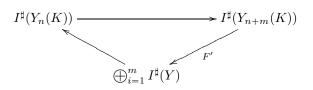

where the map F is related to the spin^c decomposition of $\widehat{HF}(W'_n(K))$.

Baldwin and Sivek constructed an analog of the term (1) in instanton theory when $b_1(W_n(K)) = 0$. The assumption of b_1 is due to the proof of some structure theorem for the cobordism map. If $b_1 \geqslant 1$, then it is harder to prove the structure theorem. Also, in their construction, the closures to define $I^\sharp(Y)$ and $I^\sharp(Y_n(K))$ are special (the connected sum with T^3). It is unknown how to extend the decomposition of the cobordism map to general closures of balanced sutured manifolds.

For the term (2), we can still use the lifts of two spectral sequences to recover filtrations. However, without the decomposition of the cobordism map, it is impossible to write down a precise statement.

For term (3), we expect that the proof [BD95, Sca15] of the usual exact triangle between $I^\sharp(Y), I^\sharp(Y_n(K))$, and $I^\sharp(Y_{n+1}(K))$ can be applied to the generalized triangle with some modifications.

Conjecture 8.1. Consider manifolds defined above. For any integer n and any positive integer m, there is an exact triangle

$$I^\sharp(Y_n(K)) \xrightarrow{\hspace{4cm}} I^\sharp(Y_{n+m}(K))$$
$$\bigoplus_{i=1}^m I^\sharp(Y)$$

with a map labeled F'.

where the map F is related to the cobordism $W_n'(K)$.

Second, for any quasi-alternating knot $K \subset S^3$, Petkova [Pet13, Section 3] proved that the chain complex $CFK^-(S^3, K)$ is determined by $\Delta_K(t)$ and the signature $\sigma(K)$. The essential observation is that in this case, $CFK^-(S^3, K)$ is chain homotopic to

$$(\widehat{HFK}(S^3, K) \otimes \mathbb{F}_2[U], \partial_z + U\partial_w).$$

where ∂_z and ∂_w shift the Alexander grading only by one. Then the result follows from the equation $\partial_z \circ \partial_w = \partial_w \circ \partial_z$ and algebraic lemmas. We can regard d_+ and d_- on $\underline{KHI}(-S^3, K)$ as analogs of ∂_w and ∂_z in instanton theory, respectively. If the following conjecture was proven, then we could apply algebraic lemmas in [Pet13, Section 3] to determine the differentials d_+ and d_- by $\Delta_K(t)$ and $\sigma(K)$. By the large surgery formula, we could compute $I^\sharp(-S_{-n}(K))$ for $|n| \geqslant 2g(K) + 1$. By results in [BS20a], we might calculate $I^\sharp(-S_r(K))$ for any quasi-alternating knot, which generalizes Theorem 1.20.

Conjecture 8.2. Suppose $K \subset S^3$ is a quasi-alternating knot and suppose the maps d_+ and d_- are on $\underline{KHI}(-S^3, K)$. Then the maps shift the grading associated to the Seifert surface by one, and the following equation holds

$$d_+ \circ d_- \doteq d_- \circ d_+,$$

where $\doteq$ means it holds up to multiplication by a unit in $\mathbb{C}$.

REFERENCES

[ABDS20] Antonio Alfieri, John A. Baldwin, Irving Dai, and Steven Sivek. Instanton Floer homology of almost-rational plumbings. *ArXiv: 2010.03800, v1*, 2020.

[AM90] Selman Akbulut and John D. McCarthy. *Casson's invariant for oriented homology 3-spheres*, volume 36 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1990.

[Bak14] Kenneth L. Baker. The Poincaré homology sphere and almost-simple knots in lens spaces. *Proc. Amer. Math. Soc.*, 142(3):1071–1074, 2014.

[BD95] Peter J. Braam and Simon K. Donaldson. Floer's work on instanton homology, knots and surgery. In *The Floer memorial volume*, volume 133 of *Progress in Mathematics*, pages 195—256. Birkhuser, Basel, 1995.

[Ber18] John Berge. Some knots with surgeries yielding lens spaces. *ArXiv:1802.09722, v1*, 2018.

[BGW13] Steven Boyer, Cameron McA. Gordon, and Liam Watson. On L-spaces and left-orderable fundamental groups. *Math. Ann.*, 356:1213–1245, 2013.

[BH20] Kenneth L. Baker and Neil R. Hoffman. The Poincaré homology sphere, lens space surgeries, and some knots with tunnel number two. *Pac. J. Math.*, 305(1):1–27, 2020.

[BM18] Kenneth L. Baker and Allison H. Moore. Montesinos knots, Hopf plumbings, and L-space surgeries. *J. Math. Soc. Japan*, 70(1):95–110, 2018.

[BM19] Kenneth L. Baker and Kimihiko Motegi. Twist families of L-space knots, their genera, and Seifert surgeries. *Commun. Anal. Geom.*, 27:743–790, 2019.

[Boa99] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures*, volume 239 of *Contemporary Mathematics*, page 49–84. American Mathematical Society, Providence, RI, 1999.

[BS15] John A. Baldwin and Steven Sivek. Naturality in sutured monopole and instanton homology. *J. Differ. Geom.*, 100(3):395–480, 2015.

[BS16] John A. Baldwin and Steven Sivek. Instanton Floer homology and contact structures. *Selecta Math. (N.S.)*, 22(2):939–978, 2016.

[BS18a] John A. Baldwin and Steven Sivek. Khovanov homology detects the trefoils. *ArXiv:1801.07634, v1*, 2018.

[BS18b] John A. Baldwin and Steven Sivek. Stein fillings and SU(2) representations. *Geom. Topol.*, 22(7):4307–4380, 2018.

[BS19] John A. Baldwin and Steven Sivek. Instanton and L-space surgeries. *ArXiv:1910.13374, v1*, 2019.

[BS20a] John A. Baldwin and Steven Sivek. Framed instanton homology and concordance. *ArXiv:2004.08699, v2*, 2020.

[BS20b] John A. Baldwin and Steven Sivek. On the equivalence of contact invariants in sutured Floer homology theories. *ArXiv:1601.04973, v3*, 2020.

[BS21] John A. Baldwin and Steven Sivek. Instanton L-spaces and splicing. *ArXiv:2103.08087, v1*, 2021.

[BVV18] John A. Baldwin and David Shea Vela-Vick. A note on the knot Floer homology of fibered knots. *Algebr. Geom. Topol.*, 18:3669–3960, 2018.

[CCG+94] Daryl Cooper, Marc Culler, Henri Gillet, Darren D. Long, and Peter B. Shalen. Plane curves associated to character varieties of 3-manifolds. *Invent. Math.*, 118(1):47–84, 1994.

[Gar19] Mike Gartner. Projective naturality in Heegaard Floer homology. *ArXiv: 1908.06237, v1*, 2019.

[Ghi06] Paolo Ghiggini. Infinitely many universally tight contact manifolds with trivial Ozsváth-Szabó contact invariants. *Geom. Topol.*, 10:335–357, 2006.

[GHVHM08] Paolo Ghiggini, Ko Honda, and Jeremy Van Horn-Morris. The vanishing of the contact invariant in the presence of torsion. *ArXiv:0706.1602, v2*, 2008.

[GL19] Sudipta Ghosh and Zhenkun Li. Decomposing sutured monopole and instanton Floer homologies. *ArXiv:1910.10842, v2*, 2019.

[GLW19] Sudipta Ghosh, Zhenkun Li, and C.-M. Michael Wong. Tau invariants in monopole and instanton theories. *ArXiv:1910.01758, v3*, 2019.

[Han20] Jonathan Hanselman. Heegaard Floer homology and cosmetic surgeries in S^3. *ArXiv: 1906.06773, v2*, 2020.

[Hed11] Matthew Hedden. On Floer homology and the Berge conjecture on knots admitting lens space surgeries. *Trans. Am. Math. Soc.*, 363(2):949–968, 2011.

[HKM09] Ko Honda, William H. Kazez, and Gordana Matić. The contact invariant in sutured Floer homology. *Invent. Math.*, 176(3):637–676, 2009.

[Hon00] Ko Honda. On the classification of tight contact structures I. *Geom. Topol.*, 4:309–368, 2000.

[JTZ18] András Juhász, Dylan P. Thurston, and Ian Zemke. Naturality and mapping class groups in Heegaard Floer homology. *ArXiv:1210.4996, v4*, 2018.

[Juh06] András Juhász. Holomorphic discs and sutured manifolds. *Algebr. Geom. Topol.*, 6:1429–1457, 2006.

[KL08] Bruce Kleiner and John Lott. Notes on Perelman's papers. *Geom. Topol.*, 12(5):2587–2855, 2008.

[KM04a] Peter B. Kronheimer and Tomasz S. Mrowka. Dehn surgery, the fundamental group and SU(2). *Math. Res. Lett.*, 11(5-6):741–754, 2004.

[KM04b] Peter B. Kronheimer and Tomasz S. Mrowka. Witten's conjecture and property P. *Geom. Topol.*, 8:295–310, 2004.

[KM10a] Peter B. Kronheimer and Tomasz S. Mrowka. Instanton Floer homology and the Alexander polynomial. *Algebr. Geom. Topol.*, 10(3):1715–1738, 2010.

[KM10b] Peter B. Kronheimer and Tomasz S. Mrowka. Knots, sutures, and excision. *J. Differ. Geom.*, 84(2):301–364, 2010.

[KM11] Peter B. Kronheimer and Tomasz S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, 113:97–208, 2011.

[Li18a] Zhenkun Li. Contact structures, excisions, and sutured monopole Floer homology. *ArXiv:1811.11634*, 2018.

[Li18b] Zhenkun Li. Gluing maps and cobordism maps for sutured monopole Floer homology. *ArXiv:1810.13071, v3*, 2018.

[Li19] Zhenkun Li. Knot homologies in monopole and instanton theories via sutures. *ArXiv:1901.06679, v6*, 2019.

[Lim09] Yuhan Lim. Instanton homology and the Alexander polynomial. *ArXiv:0907.4185, v2*, 2009.

[Lin92] Xiao-Song Lin. A knot invariant via representation spaces. *J. Differ. Geom.*, 35(2):337–357, 1992.

[Lin16] Jianfeng Lin. SU(2)-cyclic surgeries on knots. *Int. Math. Res. Not.*, 19:6018–6033, 2016.

[LL21] Zhenkun Li and Yi Liang. Some computations on instanton knot homology. *ArXiv:2102.07300, v1*, 2021.

[LM16] Tye Lidman and Allison H. Moore. Pretzel knots with L-Space surgeries. *Michigan Math. J.*, 65:105–130, 2016.

[LM21] Charles Livingston and Allison H. Moore. Knotinfo: Table of knot invariants. URL: `knotinfo.math.indiana.edu`, 2021.

[LPCS20] Tye Lidman, Juanita Pinzón-Caicedo, and Christopher Scaduto. Framed instanton homology of surgeries on L-space knots. *ArXiv:2003.03329, v1*, 2020.

[LPCZ21] Tye Lidman, Juanita Pinzón-Caicedo, and Raphael Zentner. Toroidal homology spheres and SU(2)-representations. *ArXiv:2101.02621, v1*, 2021.

[LV21] Christine Ruey Shan Lee and Faramarz Vafaee. On 3-braids and L-space knots. *Geom. Dedicata*, 213:513–521, 2021.

[LY20] Zhenkun Li and Fan Ye. Instanton Floer homology, sutures, and Heegaard diagrams. *ArXiv: 2010.07836, v2*, 2020.

[LY21a] Zhenkun Li and Fan Ye. An enhanced Euler characteristic of sutured instanton homology. *ArXiv:2107.10490, v1*, 2021.

[LY21b] Zhenkun Li and Fan Ye. Instanton Floer homology, sutures, and Euler characteristics. *ArXiv: 2011.09424, v2*, 2021.

[Mas12] Patrick Massot. Infinitely many universally tight torsion free contact structures with vanishing Ozsváth-Szabó contact invariants. *Math. Ann.*, 353:1351–1376, 2012.

[Mat13] Daniel V. Mathew. Sutured TQFT, torsion, and tori. *Int. J. Math.*, 24(5):1350039, 2013.

[Mor06] Hugh R. Morton. The Alexander polynomial of a torus knot with twists. *J. Knot Theory Ramif.*, 15(8):1037–1047, 2006.

[Mos71] Louise Moser. Elementary surgery along a torus knot. *Pacific J. Math.*, 38:737–745, 1971.

[Mot16] Kimihiko Motegi. L-space surgery and twisting operation. *Algebr. Geom. Topol.*, 16:1727–1772, 2016.

[OS04a] Peter S. Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004.

[OS04b] Peter S. Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.

[OS05a] Peter S. Ozsváth and Zoltán Szabó. Heegaard Floer homology and contact structures. *Duke Math. J.*, 129:39–61, 2005.

[OS05b] Peter S. Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44:1281–1300, 2005.

[OS05c] Peter S. Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[OS08] Peter S. Ozsváth and Zoltán Szabó. Knot Floer homology and integer surgeries. *Algebr. Geom. Topol.*, 8(1):101–153, 2008.

[OS11] Peter S. Ozsváth and Zoltán Szabó. Knot Floer homology and rational surgeries. *Algebr. Geom. Topol.*, 11(1):1–68, 2011.

[Pet13] Ina Petkova. Cables of thin knots and bordered Heegaard Floer homology. *Quantum Topol.*, 4(4):377–409, 2013.

[Ras03] Jacob Rasmussen. Floer homology and knot complements. *ArXiv:math/0306378, v1*, 2003.

[Ras07] Jacob Rasmussen. Lens space surgeries and L-space homology spheres. *ArXiv:0710.2531, v1*, 2007.

[Rol90] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[RR17] Jacob Rasmussen and Sarah Dean Rasmussen. Floer simple manifolds and L-space intervals. *Adv. Math.*, 322:738–805, 2017.

[Sca15] Christopher Scaduto. Instantons and odd Khovanov homology. *J. Topol.*, 8(3):744–810, 2015.

[SZ20] Steven Sivek and Raphael Zentner. SU(2)-cyclic surgeries and the pillowcase. *ArXiv: 1710.01957, v2*, 2020.

[SZ21] Steven Sivek and Raphael Zentner. A menagerie of SU(2)-cyclic 3-manifolds. *Int. Math. Res. Not.*, 2021.

[Tan09] Motoo Tange. Lens spaces given from L-Space homology 3-spheres. *Experiment. Math.*, 18(3):285–305, 2009.

[Vaf15] Faramarz Vafaee. On the knot Floer homology of twisted torus knots. *Int. Math. Res. Not.*, 2015(15):6516–6537, 2015.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[XZ21] Yi Xie and Boyu Zhang. On meridian-traceless su(2)-representations of link groups. *arXiv:2104.04839, v2*, 2021.

[Zen17] Raphael Zentner. A class of knots with simple SU(2)-representations. *Selecta Math. (N.S.)*, 23(3):2219–2242, 2017.

[Zen18] Raphael Zentner. Integer homology 3-spheres admit irreducible representations in $SL(2, \mathbb{C})$. *Duke Math. J.*, 167(9):1643–1712, 2018.

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